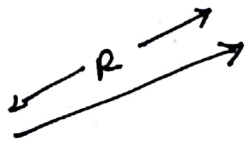
MODULE-3

# CHAPTER 4 - MAXWELL'S EQUATIONS.

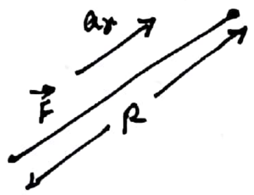
## Vector :

Vector is a quantity that has both magnitude and direction. It is typically represented by an arrow whose direction is the same as that of the quantity and whose length is proportional to the quantity's magnitude.



## Unit vector :

Unit vectors are vectors, whose magnitude is exactly 1 unit.



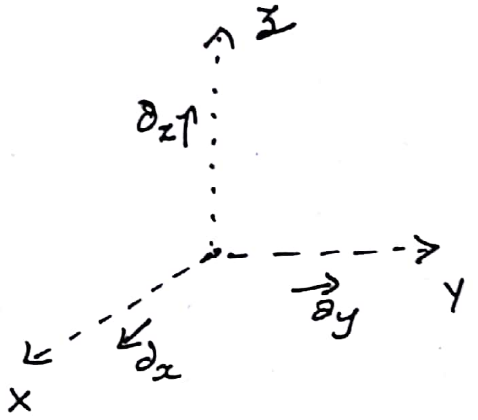
Thus, diagramatically given a vector $\vec{F}$, its magnitude is R, the distance between two points, and its direction is assigned to unit vector $\vec{a_r}$ (fig). Mathematically both the direction and magnitude $\vec{F}$ is given by the product as,

$$\vec{F} = R\vec{a_r}.$$

## Base vectors :

Base vectors are same as unit vectors,

but oriented strictly along the co-ordinates in the given co-ordinate system and pointing away from the origin.



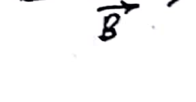

In rectangular co-ordinate system we can represent the base vectors as $a_x$, $a_y$ and $a_z$ along the x, y and z co-ordinates. (fig).

## Fundamentals of Vector Calculus

### Scalar product or Dot product:

The scalar product or dot product of two vectors is defined as the product of their magnitudes and the cosine of the smaller angle between them.

If $\vec{A}$ and $\vec{B}$ are two vectors inclined at an angle '$\theta$', their dot product is given as,




$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta.$$

### Cross product or Vector product

Cross product of two vectors is

defined as the product of the magnitudes of the two vectors and the sine of the angle between them.

Cross product of the vectors $\vec{A}$ and $\vec{B}$ is a single vector $\vec{C}$ whose magnitude is equal to the product of the magnitude of $\vec{A}$ and the magnitude of $\vec{B}$ multiplied by the sine of the smaller angle between them. The direction of $\vec{C}$ is perpendicular to the plane which has both $\vec{A}$ and $\vec{B}$ such that $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a right handed system.


$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta$$

$$\vec{A} \times \vec{B} = \vec{C}$$

In terms of the components of $\vec{A}$ and $\vec{B}$, the vector $\vec{C}$ can be expressed as a third order determinant expressed as,

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector Operator '$\nabla$'

$\nabla$ is a mathematical operator. It is called del. If it is a Cartesian co-ordinate problem, then the operation is as per the equation given below.

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

where $\hat{a_x}$, $\hat{a_y}$ and $\hat{a_z}$ are the base vectors.

Gradient, and $\nabla$

Unlike field, potential doesn't have a direction. In certain region of space, if every point in it is at the same electric potential, then there is no electric field. If there is a difference of potential between any two points in the region, an electric field does exist between them. The actual direction of the field will be in the direction in which maximum decrease of potential is established. The relation is,

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{a}_x$$

When expressed in terms of cartesian co-ordinates,

$$\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right]$$

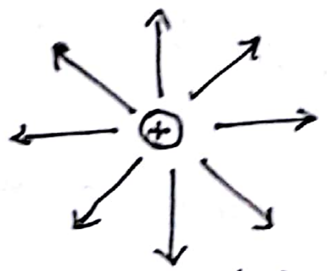
$$= -\left[\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right]V.$$

Thus, $\vec{E} = -\nabla V$ (grad V)

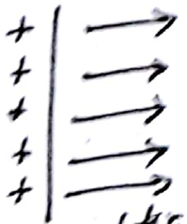
## Divergence and Curl

The vector operations $\nabla \cdot \vec{A}$ and $\nabla \times \vec{A}$ are called divergence and curl. These two operations can be carried out only in a vector field.

### Convection for Directions of field in Electrostatics.



(fig i)
Field lines due to +ve charge



(fig ii)
Field due to a +vely charged plane.

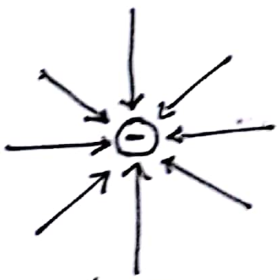


fig (iii)
Field lines due to a -ve charge

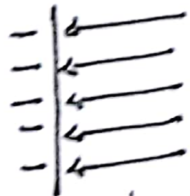


fig iv
Field lines due to a -vely charged plane.

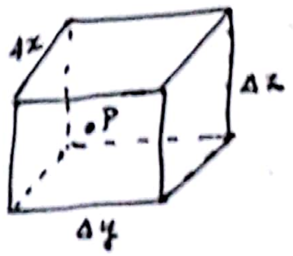
In electrostatics, the field diverges radially from a positive charge (fig i). For a positively charged plane, the field points away from the plane (fig ii). In case of negative charges, it is just the opposite. fig(iii) and iv.

The direction of the field line at any given point is the direction along which a positive charge would experience the force when placed in the field at that point.

Divergence, $\nabla \cdot \vec{A}$

The divergence of a vector field $\vec{A}$ at a given point P means, it is the outward flux per unit volume

Consider an elementary volume $\Delta v$ around a point P, the divergence at 'P' can be represented as,




Elementary volume around 'P'

$$\text{div}\,\vec{A} = \lim_{\Delta v \to 0} \frac{\text{outward flux of } \vec{A}}{\Delta v}$$

Mathematically we can write it as,

$$\text{div}\,\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot ds}{\Delta v} \quad \text{— (1)}$$

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Now, by considering a rectangular parallelopiped around the given point P as the elementary volume $\Delta v$ and working the total outward flux from all its six faces, it is possible to show that

$$\lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot d\vec{s}}{\Delta v} = \left. \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right|_{\text{at } P.}$$

From eqn ①,

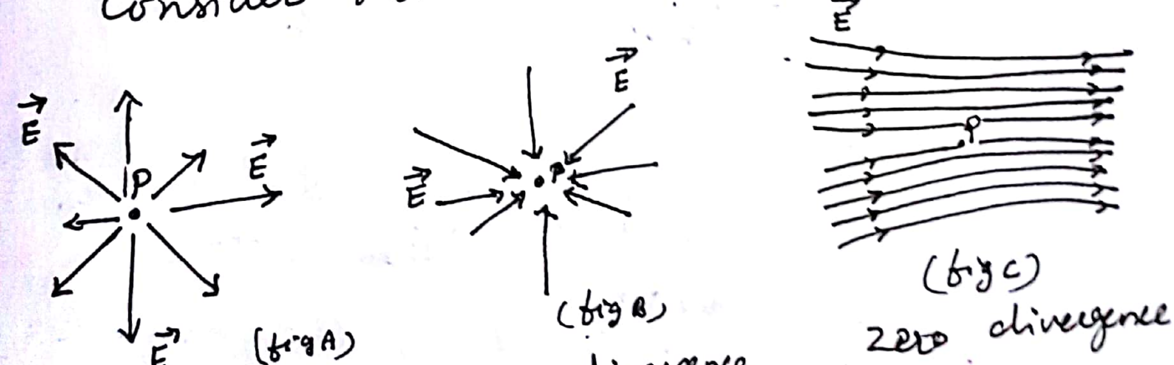
$$\text{divergence of } \vec{A} = \left[ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right]$$

or, $\text{divergence of } \vec{A} = \nabla \cdot \vec{A}.$

Physical Significance of Divergence

If there are positive charges densely packed at a point, then a large no. of field lines diverge from that point. If there are negative charges, the vectors converge at that point.

Consider three cases as shown in figure.




(fig A) +ve charge

(fig B) −ve divergence

(fig C) zero divergence.

Let the field be an electric field $\vec{E}$. In fig A there is a point 'P' from which the field $\vec{E}$ vectors diverge. It indicates a source of positive charges at 'P'. The divergence at P is positive. In fig (B) since the vectors converge, it is a negative divergence indicating the presence of negative charges at P. In fig (C), exactly the same number of vectors both converge and diverge at P. Hence it is zero divergence.

A vector field whose divergence is zero is called solenoidal field.

## Curl $\nabla \times \vec{A}$

Curl $\vec{A}$ is represented as a vector whose direction is normal to the area around a pt, 'P' when the area is oriented to make the circulation maximum.

It can be represented as,

$$\text{Curl } \vec{A} = \lim_{\Delta S \to 0} \frac{\text{Max. circulation around } P}{\Delta S}$$

Mathematically,

$$\text{Curl } \vec{A} = \left[ \lim_{\Delta S \to 0} \frac{\oint \vec{A} \cdot dl}{\Delta S} \right] \hat{a}_n$$

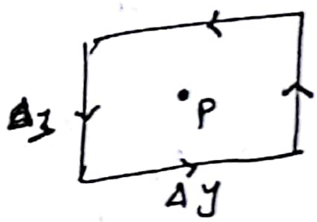
Where, $\Delta S$ is the elementary area bounded by the curve $L = \oint dl$.

$\hat{a}_n$ is the unit vector normal to $\Delta S$.

Now, by considering a rectangular elementary area across the point P as $\Delta S$ and working the closed line integral about the 4 sides of the boundary line, it is possible to show that.

$$\left[\lim_{\Delta S \to 0} \frac{\oint \vec{A}\cdot d\vec{l}}{\Delta S}\right] \hat{a}_n = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

where the right side is the third order determinant. Since the left side is the curl of A, we can write,



Rectangular contour around P.

$$\text{Curl of } \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

The right side is equal to $\nabla \times \vec{A}$

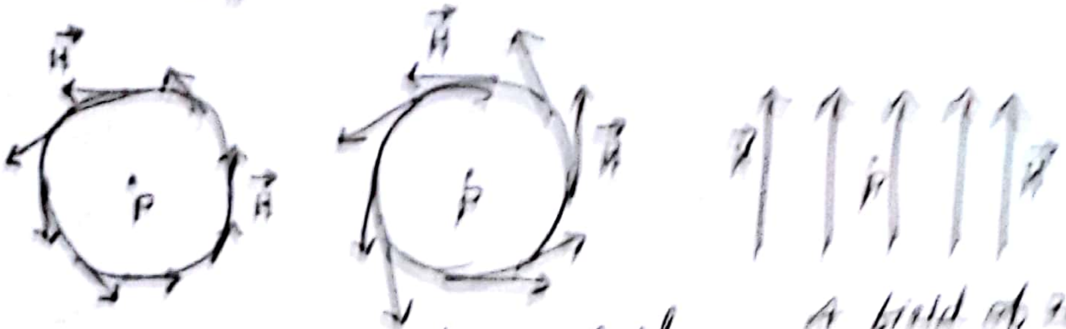
$$\therefore \text{Curl of } \vec{A} = \nabla \times \vec{A}$$

Physical significance of curl.

The curl of a vector field $\vec{A}$ at a point P is a measure of how much the field curls around P.

Now let the field be a magnetic field $\vec{H}$ around the point P.

Consider fig (A), (B), and (C)



Curl of $\vec{H}$ fig (A)

Larger curl fig (B)

A field of zero curl fig (C)

In fig A, the magnetic field vectors curl around P. When a current is passed through a straight conductor, the magnetic field curls around the conductor in the same way.

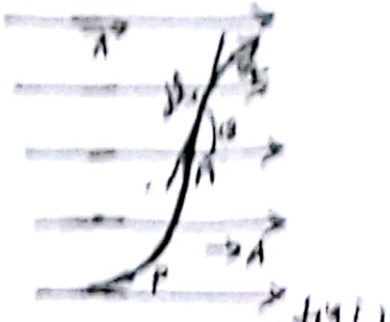
In fig B, the magnetic field vectors have larger magnitude all around P.

In fig C, the field vectors though have larger magnitude, do not have any turning motion at all. In this case $\nabla \times \vec{H}$ turns to be zero, meaning no curl.

A vector field whose curl is zero is called irrotational.

Linear, Surface, and Volume Integrals

Linear integral ( Line integral)



(fig 1)

Consider a linear path of length 'L' from P to Q in a vector field $\vec{A}$ (fig 1). The line consists of small lengths $dl$. Consider one such element at 'M'. At M draw a tangent. The tangent make an angle 'θ' with $\vec{A}$. Then we have,

$$\vec{A} \cdot \vec{dl} = A\, dl \cos\theta$$

Integrate,

$$\int_P^Q \vec{A} \cdot \vec{dl} \quad \text{--- (1)}$$

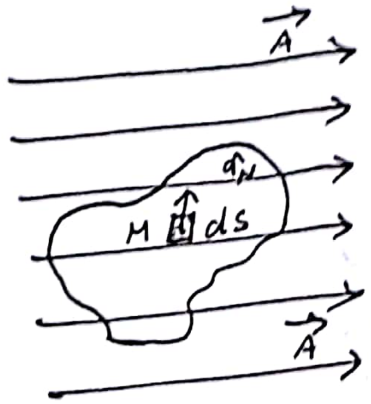
This is the line integral of path PQ.

If the path is a closed curve of length 'L' as shown in figure. Then eqn (1) becomes a closed contour integral denoted as $\oint$.

ie, Line Integral $= \oint_L \vec{A} \cdot \vec{dl}$ --- (2)

Closed path in a field.

# Surface Integral




Consider a surface of area 'S' (fig) in a vector field $\vec{A}$. The surface could be thought of as made up of a number of elementary surfaces each of area ds. Let $\hat{a}_N$ be the unit vector normal to ds at M. In a vector field the elementary surface ds acts as a vector $\vec{ds}$ given as,

$$\vec{ds} = ds \cdot \hat{a}_N .$$

If we take the dot product $\vec{A} \cdot \vec{ds}$, it represents the flux of the vector field $\vec{A}$ through $\vec{ds}$. The flux of the field $\vec{A}$ through the surface 'S' can be obtained by integrating $\vec{A} \cdot \vec{ds}$ over the entire surface S.

$$\psi = \int_S \vec{A} \cdot \vec{ds} \quad \text{—— (3)}$$

'$\int_S$' is the symbol for surface integral.

If the surface is a closed one,

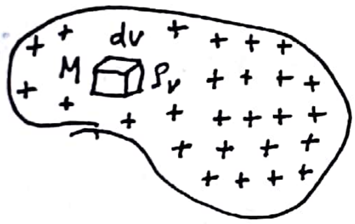
The total flux '$\psi$' in equation (3) becomes,

$$\psi = \oint_S \vec{A} \cdot \vec{ds} \quad \text{—} (4)$$

$\oint_s$ is the symbol for closed surface integral.

Also, $\psi = \oint_S \left[\vec{A} \cdot \hat{a}_N\right] ds$ —— (5)

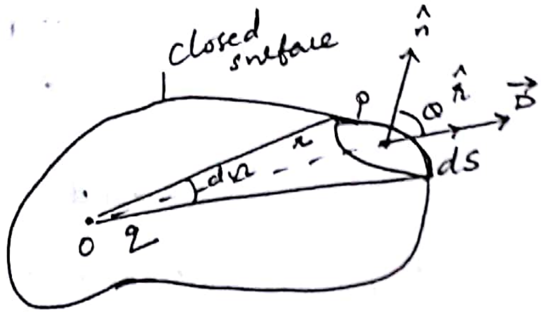
## Volume Integral




If the charges are distributed continuously in a volume, then it is referred to as volume charge distribution. Consider an elementary volume dv at M. Let the charge density at M be $\rho_v$. Then the volume integral of $\rho_v$ over the volume v is $\oint_V \rho_v \, dv$. Here $\oint_V$ is the symbol for volume integral.

## Gauss' law in electrostatics

Consider electric charges in a certain region. A closed surface of any shape can be imagined surrounding these charges (fig). Such a surface is called "Gaussian Surface".




Flux through closed surface.

Consider a charge 'q' and a closed surface surrounding it. This closed surface is made up of a number of elementary surfaces each of area $dS$. If $\vec{D}$ is the flux density at $ds$, then it can be shown that the total electric flux over the entire surface.

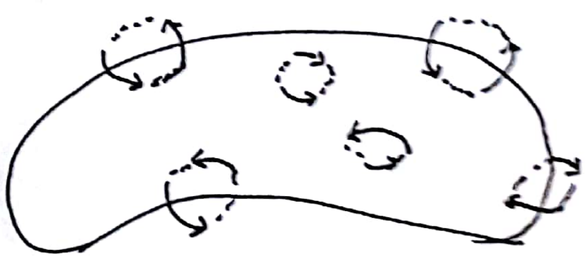
$$\psi = \oint_S \vec{D} \cdot \vec{ds} = q.$$

If there are number of charges, $q_1, q_2 \dots$ inside the surface, then,

$$\psi = \oint \vec{D} \cdot \vec{ds} = \Sigma q.$$

or, $\psi = Q$ the total charge enclosed.

Gauss' Law for Magnetic Fields and Maxwell's Equation in Magnetostatics



Closed Surface in a magnetic field

Let us consider a closed surface of any shape in a magnetic field. The property of the magnetic field is that a magnetic flux line always follows a closed loop.

Thus for a closed surface in a magnetic field, for every flux that enters in to the surface, there will always a flux line coming out of the surface. Then the flux for the entire closed surface,

total outward flux = total inward flux.

But the outward flux is taken as +ve and the inward flux is negative

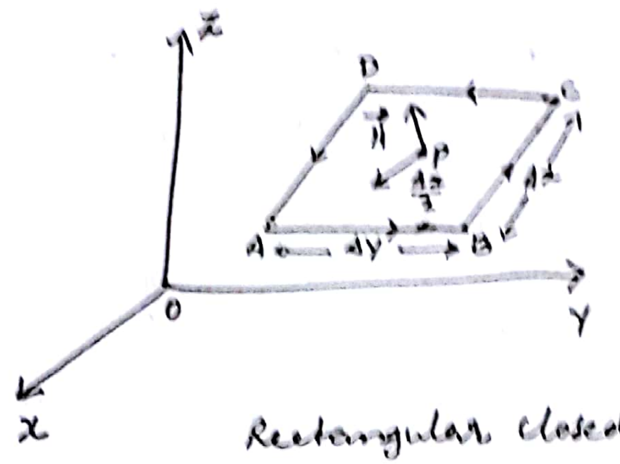
$\therefore$ The total flux summed over the entire Gaussian surface = 0. In vector form, $\nabla \cdot \vec{B} = 0$ — (1)

where $\vec{B}$ is the magnetic flux density. This is one of the four Maxwell's equations.

<u>Current density</u> $\vec{J}$

It is a vector which is related to the electric current. It is defined as the current per unit area of cross-section of an imaginary plane held normal to the direction of current in a current carrying conductor.

Ampere's law




Rectangular closed path in a field

Consider a point 'p' in a magnetic field $\vec{H}$ (fig). Imagine a rectangular loop ABCD around 'p' in a plane parallel to X-Y plane. If $J_z$ and $\hat{a}_z$ are the current density and unit vector respectively along z-direction. Then the line integral of $\vec{H} \cdot \vec{dL}$ over the closed path ABCDA is,

$$\left[\text{Curl } \vec{H}\right]_1 = J_z \hat{a}_z$$

The curl is in a plane parallel to x-y plane.

11ly for the curl in a plane parallel to y-z plane we have,

$$\left[\text{Curl } \vec{H}\right]_2 = J_x \hat{a}_x \quad \text{and}$$

for the curl in a plane parallel to x-z plane,

$$\left[\text{Curl } \vec{H}\right]_3 = J_y \hat{a}_y$$

Here, $\left[\text{Curl } \vec{H}\right]_1$, $\left[\text{Curl } \vec{H}\right]_2$ and $\left[\text{Curl } \vec{H}\right]_3$ are the components of curl of $\vec{H}$ around the point P.

But curl $\vec{H}$ is represented as $\nabla \times \vec{H}$

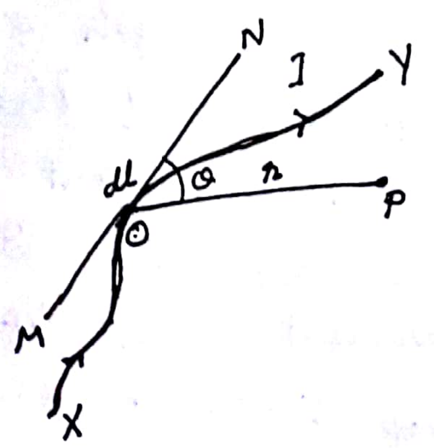
$$\nabla \times \vec{H} = J_x \hat{a_x} + J_y \hat{a_y} + J_z \hat{a_z}$$

$$\nabla \times \vec{H} = \vec{J} \quad \text{———} \quad (2)$$

where $\vec{J}$ is the current density vector with components $J_x$, $J_y$ and $J_z$ along the 3 Cartesian co-ordinates.

Eqn (2) is the ampere's circutal law for time-invariant fields. This is another Maxwell's equation.

## Biot - Savart's Law

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The Biot-Savart's law gives both the magnitude and direction of magnetic field intensity at a point due to the current in a differential element of a current carrying conductor.

Let XY be a conductor carrying a current I (fig). Consider a differential element of length dl at O on the conductor at which a tangent MN is drawn. Let 'p' be a point at a distance 'r' from 'O'. The angle between MN and OP be 'θ'.

Statement of Biot- Savart's law

As per Biot- Savart's law, the magnitude of the magnetic field intensity $d\vec{H}$ at a point due to the current in the differential element is directly proportional to the product of the current I the magnitude of the length of the differential element dl, and the sine of the angle between the tangent drawn to the element and the line joining the point and the element (ie, sinθ) and it is inversely proportional to the square of the distance between the point and the element :- (ie, $\frac{1}{r^2}$).

$$dH \propto \frac{I\,dl\,\sin\theta}{r^2}$$

$$dH = K\,\frac{I\,dl\,\sin\theta}{r^2}$$

where K is the proportionality constant,

$$K = \frac{1}{4\pi}$$

$$\therefore dH = \frac{I\,dl\,\sin\theta}{4\pi r^2}$$

In vector notation, the Biot-Savart's law is,

$$d\vec{H} = \frac{I\,d\vec{l} \times \hat{a}_r}{4\pi r^2}$$

where, the vector $d\vec{l}$ is directed along the direction of the current and $\hat{a}_r$ is the unit vector directed from the point 'O' towards P.

Faraday's law

(Faraday's law of Electromagnetic Induction)

Faraday's law states that the magnitude of the induced emf in a circuit is equal to the rate of change of magnetic flux through it, and its direction opposes the flux change

The induced emf 'e' is expressed as,

$$e = -\frac{d\phi}{dt} \quad \text{— ①}$$

where '$\phi$' is the flux linkage with the circuit.

If we consider a coil with 'N' turns, then the emf induced across the coil is given by,

$$e = -N\frac{d\phi}{dt} \quad \text{— ②}$$

Integral and Point (or differential) forms of Faraday's Law

Consider a loop of a conducting material which is stationary.


According to Faraday's law, the induced emf

$$e = -\frac{d\phi}{dt}$$

$$\frac{d\phi}{dt} = \int_s \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

The induced emf in the loop is given by,

$$e = \oint \vec{E} \cdot dl$$

$$\oint \vec{E} \cdot d\vec{L} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} \quad \text{— ③}$$

This is the Faraday's law in integral form.

By, Stoke's theorem,

$$\oint \vec{E}\cdot d\vec{l} = \int_S (\nabla \times \vec{E})\cdot d\vec{s} \quad — (4)$$

Equating R·H·S of (3) and (4)

$$\int_S (\nabla \times \vec{E})\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

$$\int_S \left[(\nabla \times \vec{E}) + \frac{\partial \vec{B}}{\partial t}\right]\cdot d\vec{s} = 0.$$

Since $d\vec{s}$ can not be zero

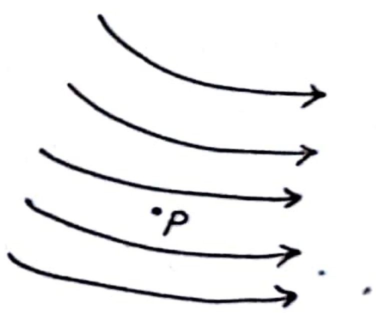
$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0.$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad —— (III)$$

This is the differential form of Faraday's law and one of Maxwell's equation.

## Gauss Divergence Theorem

Consider a point 'p' in a region where there are charges. (fig) Let '$\rho_v$' be the charge density at 'p'. Due to the charges there will be electric field around 'p'.




Point in a region of charges.

We have,
$$\rho_v = \lim_{\Delta v \to 0} \left[\frac{\Delta Q}{\Delta v}\right] = \frac{dQ}{dv} \quad \text{— (2)}$$

$$dQ = \rho_v \, dv$$

If Q is the total charge,
$$Q = \int dQ = \int_V \rho_v \, dv$$

But $\nabla \cdot \vec{D} = \rho_v$
$$Q = \int_V \nabla \cdot \vec{D} \, dv \quad \text{— (3)}$$

By applying Gauss law,
$$\oint_S \vec{D} \cdot \vec{ds} = Q \quad \text{— (4)}$$

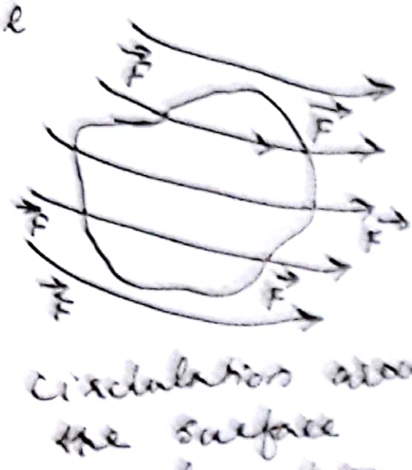
By eqn (3) and (4),
$$\oint_S \vec{D} \cdot \vec{ds} = \int_V \nabla \cdot \vec{D} \, dv \quad \text{— (5)}$$

This is the Gauss' divergence theorem.

## Stokes' Theorem

Stoke's theorem gives a way of relating a line integral to a surface integral if curl of a vector exists in a vector field.

It is given by,
$$\int \left[\nabla \times \vec{F}\right] \cdot \vec{ds} = \oint \vec{F} \cdot \vec{dl}$$




circulation around the surface boundary.

If we consider curl of a vector $\vec{F}$ at each point in a vector field, the sum of the curls will be equal to the circulation of $\vec{F}$ around the boundary of the closed surface.

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## Current density $\vec{J}$

Current density is the current per unit area of cross section. If $\rho_v$ is the charge density and $\vec{v}$ is the velocity of charge flow, then $\vec{J} = \rho_v \vec{v}$.

## Equation of Continuity


For direct current in a circuit (fig), if we consider a closed surface at some part of it, then the charge flow in to the surface is equal to the charge outflow. ie, there is no net charge within the surface.

$$\nabla \cdot \vec{J} = 0 \quad \text{—— (1)}$$

The ampere's law is expressed as,

$$\nabla \times \vec{H} = \vec{J}$$

Take the divergence on both sides,

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} \quad \text{—— (2)}$$

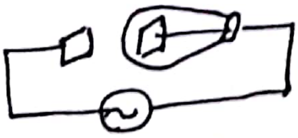
But as per vector analysis, the divergence of a curl for any field is always zero.

$$\nabla \cdot \vec{J} = 0$$

But it fails under time dependent variation.

Consider an a c circuit with a capacitor.

A closed surface encloses only one plate of the capacitor (fig). Whenever there is any current flow in to the closed surface, it is not accompanied by a current outflow through any part of the surface. Thus the condition $\nabla \cdot \vec{J} = 0$ fails in this case.



closed surface enclosing a capacitor's plate in AC circuit.

According to equation of continuity, if there is any charge ~~outfl~~ outflow through the surface, it is accompanied by a reduction of charge within the surface.

ie, $\nabla \cdot \vec{J} = -\dfrac{\partial \rho_v}{\partial t}$.

Where $\vec{J}$ is the current density and $\rho_v$ is the charge density. This equation is called equation of continuity.

## Displacement Current

According to Gauss' law

$$\nabla \cdot \vec{D} = \rho_v$$

Differenciating w.r.to time,

$$\frac{\partial}{\partial t}(\nabla\cdot\vec{D}) = \frac{\partial \rho_v}{\partial t}$$

$$\nabla\cdot\left[\frac{\partial \vec{D}}{\partial t}\right] = \frac{\partial \rho_v}{\partial t} \quad \text{--- (1)}$$

Equation of continuity is,

$$\nabla\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

$\therefore$ eqn (1) becomes,

$$\cancel{\nabla\cdot\vec{J} = \frac{\partial \rho_v}{\partial t}}$$

$$\nabla\cdot\vec{J} = -\nabla\cdot\left(\frac{\partial \vec{D}}{\partial t}\right)$$

$$\nabla\cdot\left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right] = 0$$

Now, for the time varying case, it is not $\nabla\cdot\vec{J} = 0$, but the equation

$\nabla\cdot\left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right] = 0$ must be considered.

ie, $\vec{J}$ must be replaced by $\vec{J} + \frac{\partial \vec{D}}{\partial t}$.

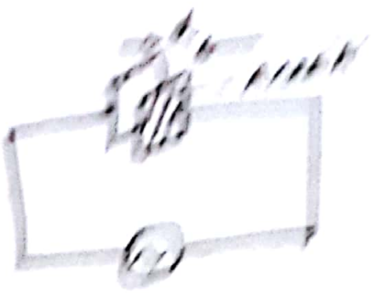
$\therefore$ Ampere's circuital law becomes,

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{--- (2)}$$

This equation is considered as Maxwell's-Ampere's law.

$\frac{\partial \vec{D}}{\partial t}$ is the displacement current density.

Expression for Displacement current



Consider a parallel plate capacitor connected across an a.c source. Let the area of each of the plate be A.

(i) Conduction Current

Let the potential be $V = V_0 e^{i\omega t}$

$D = \varepsilon E$

For a parallel plate capacitor, $E = \frac{V}{d}$

$$D = \frac{\varepsilon V}{d} = \frac{\varepsilon}{d} V_0 e^{i\omega t}$$

The displacement current density is, $\left(\frac{\partial D}{\partial t}\right)$

If $I_D$ is the displacement current,

then, $\left[\frac{\partial D}{\partial t}\right] = \frac{I_D}{A}$

$$I_D = \left[\frac{\partial D}{\partial t}\right] A = \frac{\partial}{\partial t}\left[\frac{\varepsilon}{d} V_0 e^{i\omega t}\right] A$$

$$\text{ie } I_D = \frac{j\omega \varepsilon A}{d} V_0 e^{i\omega t} \quad \text{--- (3)}$$

Displacement current is the correction factor in Maxwell's equation that appears in time varying condition but doesn't describe any movement of charges.

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## MAXWELL'S EQUATIONS

Time-varying fields (Differential form)

1. From Gauss' law in electrostatics,

$$\nabla \cdot \vec{D} = \rho_v \quad \text{— I}$$

2. From Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \quad \text{— II}$$

3. From Gauss' law form magnetic fields.

$$\nabla \cdot \vec{B} = 0. \quad \text{— III}$$

4. From Ampere's law

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{— IV}$$

In case of static fields, (time independent fields)

1. $\nabla \cdot \vec{D} = \rho_v$
2. $\nabla \times \vec{E} = 0$
3. $\nabla \cdot \vec{B} = 0$
4. $\nabla \times \vec{H} = \vec{J}$

## ELECTROMAGNETIC WAVES

✓ Wave equations in differential form in free space in terms of field

According to Maxwell's eqns;

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

But, $D = \epsilon E$ and $B = \mu H$

$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad — (1)$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad — (2)$$

Taking curl for both sides of eqn (2), we have,

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H}) \quad — (3)$$

As per vector analysis,

$$\nabla \times \nabla \times \vec{E} = \nabla (\nabla \cdot \vec{E}) - \nabla^2 E$$

$$= \nabla \left( \frac{\rho_v}{\epsilon} \right) - \nabla^2 \vec{E} \quad — (4)$$

(Since $\nabla \cdot \vec{D} = \rho_v$ or $\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon}$).

From eqn (3) and (4)

$$\nabla \left( \frac{\rho_v}{\epsilon} \right) - \nabla^2 E = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H})$$

Using eqn (1),

$$\nabla \left( \frac{\rho_v}{\epsilon} \right) - \nabla^2 E = -\mu \frac{\partial}{\partial t} \left[ \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \right]$$

$$\nabla^2 E - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{J}}{\partial t} + \nabla \left( \frac{\rho_v}{\epsilon} \right) - (5)$$

Eqn (5) is the wave equation in $\vec{E}$ for a homogeneous and isotropic medium.

In free space, eq (5) becomes,

$$\nabla^2 E - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad — (6).$$

($\because \rho_v = 0$, $J = 0$).

# Plane electromagnetic waves in vacuum

Electromagnetic waves propagating along x-axis, referred to as uniform plane waves if the electric field in independent of y and z axis.

We can write the equations for plane electromagnetic waves in vacuum for ~~to~~ $\vec{E}$ and $\vec{B}$ as follows.

Let '$\lambda$' be the wavelength of a plane electromagnetic wave propagating in vacuum along +ve x direction. The electric field associated with the wave be oriented parallel to y-direction and is denoted as $E_y$. Let A be its amplitude. There will be a magnetic field associated with the wave and is oriented parallel to z-direction. Let it be denoted as $\vec{B_z}$

Then $\vec{E_y}$ and $\vec{B_z}$ at any instant 't' can be written as,

$$\vec{E_y} = A \cos\left[\frac{2\pi}{\lambda}(x - ct)\right] \hat{a}_y ,$$

$$\vec{B_z} = \frac{1}{c} A \cos\left[\frac{2\pi}{\lambda}(x - ct)\right] \hat{a}_z$$

where 'c' is the velocity of propagation of e.m wave in vacuum.

By taking the ratio of magnitudes of $\vec{E}_y$ and $\vec{B}_z$ in the above two eqns,

$$c = \frac{E_y}{B_z}$$

where $E_y = |\vec{E}_y|$ and $B_z = |\vec{B}_z|$

We can also express 'c' in terms of $\mu_0$ and $\epsilon_0$ as follows,

The classical wave equations for an oscillating physical quantity $\vec{F}$ is represented as,

$$\nabla^2 F - \frac{1}{v^2}\frac{\partial^2 \vec{F}}{\partial t^2} = 0 \quad — (4)$$

where 'v' is the velocity of light

Comparing eqn (3) and (4) $\quad \frac{1}{v^2} = \mu\epsilon$

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

In vacuum $\mu = \mu_0$ and $\epsilon = \epsilon_0$. Also, the velocity of e m wave is taken as 'c' the velocity of light.

The above equation becomes,

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

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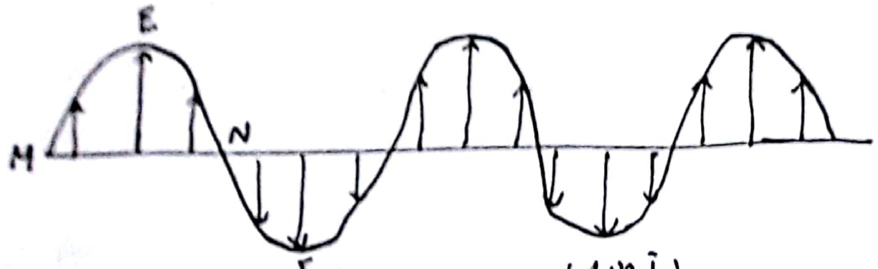
# POLARIZATION OF EM WAVES

## Representation of em waves in polarization




(fig 1)


(fig 2)

CONVENTIONAL REPRESENTATION

For the representation of polarization of EM waves, only the electric field variation is showing, assuming that the associated magnetic field variation is invariably there in the perpendicular plane. This is because, 'E' is more effective at exerting forces, or doing work on charges in comparison to that of 'B'.

From (fig 1) we see that the value of 'E' increases from zero reaches a maximum, and gradually becomes zero at N, then starts increasing in the opposite direction to reach a maximum, and again becomes zero. If we observe the variation of 'E', the tip of the 'E' vector appears to oscillate along a single straight line as shown in fig 2.

## Different Polarized Waves

There are 3 different polarized waves.

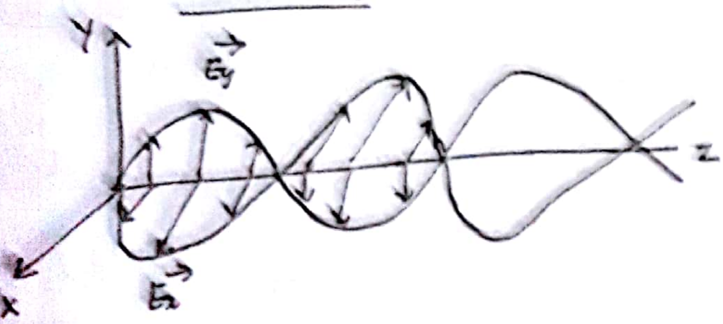
1. Linearly polarized wave
2. Circularly polarized wave and
3. Elliptically polarized wave.

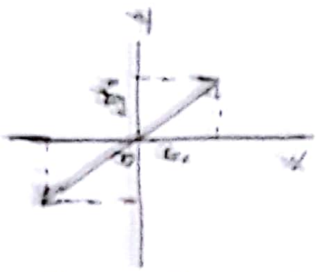
For a linearly polarized wave, if the variation of $\vec{E}$ is projected on a plane perpendicular to the direction of propagation, then the tip of the $\vec{E}$ vector in the projection traces a straight line.

If it is a circularly polarized wave, then the tip of $\vec{E}$ vector in the projection traces a circle.

For an elliptically polarized wave, the trace will be an ellipse.

## Linear Polarization




Linearly polarized wave

The electric field $\vec{E}$ is the resultant of two rectangular components $\vec{E_x}$ and $\vec{E_y}$. $\vec{E_x}$ and $\vec{E_y}$

are directed along x and y-directions, and the path of propagation is along the z-axis.

In fig(i) $\vec{E_x}$ and $\vec{E_y}$ components are in identical phase but of different amplitudes.

Since both $\vec{E_x}$ and $\vec{E_y}$ are in phase, both of them reach maxima and minima values simultaneously. The variation of $\vec{E}$ when viewed from a location to which z axis is normal, the tip of $\vec{E}$ vector traces along a straight line inclined at an angle '$\theta$' to x-direction. Thus we say that the wave is linearly polarized.

Condition for linear polarization

The condition for linear polarization for a wave propagating in z-direction is that, the $\vec{E_x}$ and $\vec{E_y}$ components must be in phase, but their amplitudes may be either equal or unequal.

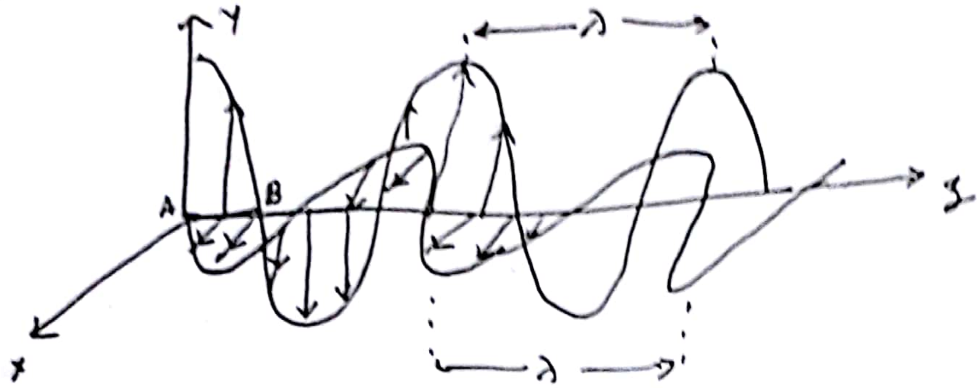
Let $E_1$ and $E_2$ be the amplitudes of $E_x$ and $E_y$ and '$\delta$' the phase difference between the two. For a linearly polarized wave $\delta = 0$

$$E_y = \left(\frac{E_2}{E_1}\right) E_x.$$ If $(E_2/E_1) = m$,

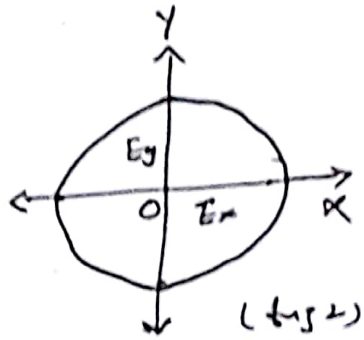
$$E_y = m E_x.$$

The above eqn is an eqn. for a st. line.

Circular Polarization




(fig 1) Circular polarization




(fig 2)
Circularly polarized wave.

Consider the two components $\vec{E_x}$ and $\vec{E_y}$ to be of equal amplitudes. ie, $E_1 = E_2$. But let one of them. $\vec{E_y}$ be ahead of $\vec{E_x}$ by a phase of 90°, as shown in fig (1).
The resultant field $\vec{E}$ tilt progressively towards horizontal direction from A to B. For an observation along the z axis, this is continuous rotation of $\vec{E}$ vector without changing its magnitude from vertical to horizontal position. The $\vec{E}$ vector continuous its rotation, and completes one cycle of rotation in a span of one wavelength. Thus the tip of $\vec{E}$ vector in the projection traces a circle.

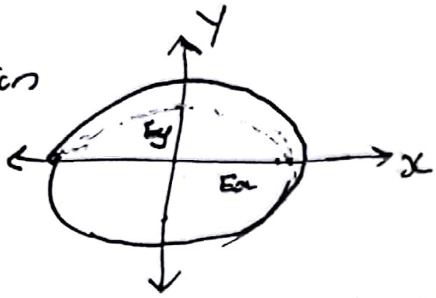
Condition for circular polarization

The condition for circular polarization

for a wave travelling in z direction is that, the component, $\vec{E_x}$ and $\vec{E_y}$ must have a constant phase difference of 90° and their amplitudes are equal.

For circular polarization $E_1 = E_2$ and phase difference $\delta = \frac{\pi}{2}$. For these conditions it can be shown that,
$$E_x^2 + E_y^2 = E_1^2$$
which is the equation for a circle.

## Elliptical Polarization




For elliptical polarization the amplitudes of $\vec{E_x}$ and $\vec{E_y}$ are unequal. Hence the vectorial sum of $\vec{E_x}$ and $\vec{E_y}$ produces a rotating $\vec{E}$ vector.

Elliptically polarized wave

whose magnitude varies between a maximum and a minimum. Thus the tip of the $\vec{E}$ vector traces along an ellipse on a plane perpendicular to the z axis.

Condition for Elliptical polarization

The components of $\vec{E_x}$ and $\vec{E_y}$ must have a constant non-zero phase difference and their amplitudes are unequal.

ie, $E_x \neq E_y$ (fig). $\delta \neq 0$, Let us assume $\delta = \frac{\pi}{2}$. Then
$$\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1.$$
This is ~~the~~ equation for ellipse.