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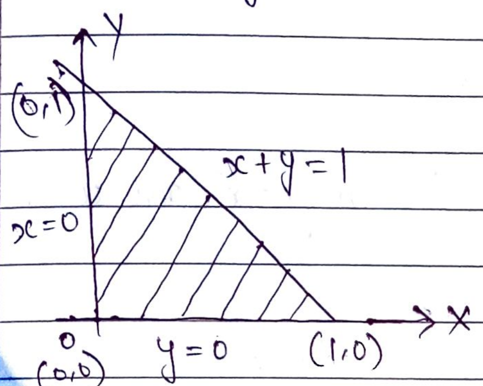
Evaluate

(#) $\iint_R xy \, dx \, dy$ where R is the region bounded by coordinate axis and the line $x+y=1$.

$$\iint_R xy \, dx \, dy$$

$x+y=1$ $\Rightarrow x = 1-y$

$\Rightarrow y = 1-x$



$x=0, \quad y=0$

$y=1, \quad x=1$

$$I = \iint_R f(x,y) \, dx \, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{y=1-x} xy \, dy \, dx$$

$$= \int_{x=0}^{1} \left[x \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 \left[x((1-x)^2 - 0) \right] dx$$

$$= \frac{1}{2} \int_0^1 x(1+x^2-2x) \, dx$$

$$= \frac{1}{2} \int_0^1 (x + x^3 - 2x^2) \, dx$$

0.535

$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3}\right]_0^1$$

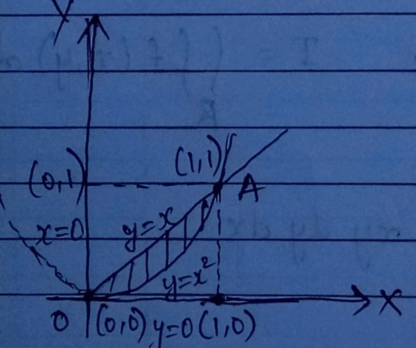
$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right]$$

$$= \frac{1}{24}$$

(#) Evaluate

$$\iint_R xy\,(x+y)\,dy\,dx$$

taken over the area between $y = x^2$ and $y = x$



$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0,\ x = 1$$

$y = x \rightarrow$ 1) $x = 0$
$\Rightarrow y = 0$

2) $x = 1$
$\Rightarrow y = 1$

$y = x^2 \longrightarrow$ 1) $x = 0$
$\Rightarrow y = 0$

2) $x = 1 \Rightarrow y = 1$

$$I = \iint_R xy\,(x+y)\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{y=x} (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left[x^2 \frac{y^2}{2} + x \frac{y^3}{3} \right]_x^{x} dx$$

$$= \int_0^1 \frac{x^2}{2}(x^2 - x^4) + \frac{x}{3}(x^3 - x^6)\, dx$$

$$= \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3} \right) dx$$

$$\left[\frac{x^5}{10} - \frac{x^4}{14} + \frac{x^5}{15} - \frac{x^8}{24} \right]_0^1$$

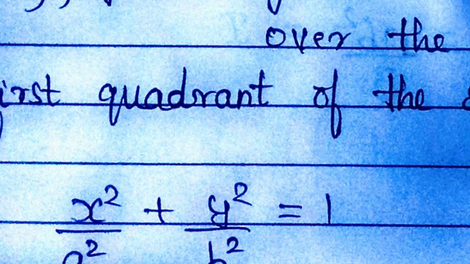
$$= \left[\frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} \right] = \frac{3}{56}$$

Evaluate

$$\iint y \, dx \, dy$$

over the region bounded by the first quadrant of the Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$I = \int_{x=0}^{a} \int_{y=0}^{y = b/a \sqrt{a^2 - x^2}} y \, dy \, dx$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2\left[\frac{a^2-x^2}{a^2}\right] \Rightarrow y = \frac{b}{a}\sqrt{a^2-x^2}$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$= \frac{b^2}{2a^2}\int_0^a (a^2-x^2)\,dx$$

$$= \frac{b^2}{2a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left[\left(a^3 - \frac{a^3}{3}\right) - 0\right]$$

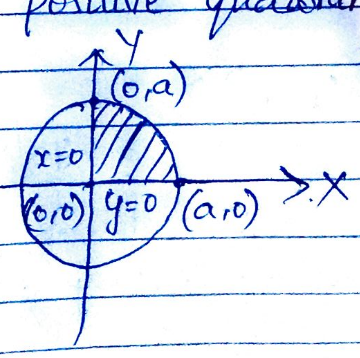
$$= \frac{b^2}{\cancel{2a^2}}\left[\frac{\cancel{2}a^{\cancel{3}}}{3}\right] = \frac{ab^2}{3} = I$$

Evaluate

$$\iint xy\,dx\,dy$$

over the positive quadrant of the circle

$x^2+y^2=a^2$



$$x^2+y^2=a^2$$
$$y^2=a^2-x^2$$
$$y=\sqrt{a^2-x^2}$$

$$I = \int_0^a \int_{y=0}^{y=\sqrt{a^2-x^2}} xy \, dy \, dx$$

$$= \int_0^a \left[x \frac{y^2}{2} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_0^a \left[x (a^2 - x^2) - 0 \right] dx$$

$$= \frac{1}{2} \int_0^a \left[xa^2 - x^3 \right] dx$$

$$= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right]$$

$$= \frac{1}{2} \left[\frac{a^4}{4} \right]$$

$$\boxed{I = \frac{a^4}{8}}$$

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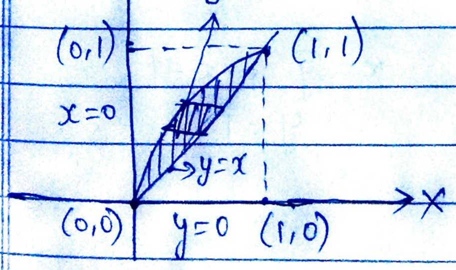
Evaluation of $\iint$ by changing the order of integration.

Evaluate : $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$ by changing order of integration.



$y = x \; ; \; y = \sqrt{x}$

$x = \sqrt{x}$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, \; x = 1$

6/12/18

1) $y = x \rightarrow x=0,\ y=0$

$x=1,\ y=1$

2) $y=\sqrt{x} \rightarrow x=0,\ y=0$

$x=1,\ y=1$

Points of Intersection $(0,0)$ $(1,1)$

$$\int_{y=0}^{y=1}\int_{x=y^2}^{x=y} xy\,dx\,dy$$

$$\int_0^1 \left[\frac{x^2 y}{2}\right]_{y^2}^{y} dy$$

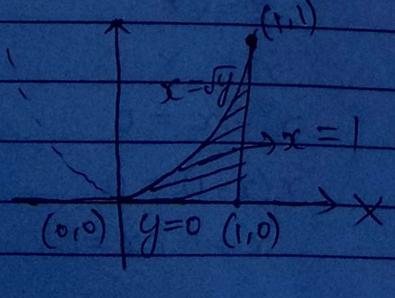
$$= \frac{1}{2}\int_0^1 y\,(y^2-y^4)\,dy$$

$$= \frac{1}{2}\int_0^1 (y^3-y^5)\,dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4}-\frac{y^6}{6}\right] = \frac{1}{2}\left[\frac{1}{4}-\frac{1}{6}\right] = \frac{1}{24}$$

(#) Change the order of integration and hence evaluate

$$\int_{y=0}^{1}\int_{x=\sqrt{y}}^{1} dx\,dy$$



$x=\sqrt{y}$, $x=1$

$y=1$

$$\int_{x=0}^{1}\int_{y=\sqrt{x}}^{1} dy\,dx$$

$$= \int_{0}^{1} dx\,[y]_{\sqrt{x}}^{1}$$

$$\int_{0}^{1} [(1-\sqrt{x})]\,dx$$

$$\left[x - \frac{x^{1/2+1}}{1/2+1}\right]_{0}^{1}$$

$$= \left[x - \frac{x^{3/2}}{3/2}\right]_{0}^{1}$$

$$2/3\left[x - (x)^{3/2}\right]_{0}^{1}$$

$$= 3/2\,[1-0] = 2/3$$

$$\boxed{\cancel{I = 2/3}} \qquad 2/3\,[3/2 - 1] = \cancel{2}\times \frac{2}{3}\times\frac{1}{2}$$

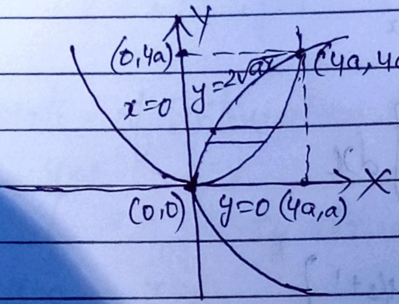
$$\boxed{I = \frac{1}{3}}$$

(#) Change the order of integration and hence evaluate

$$\int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$$

$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$



$\frac{x^2}{4a} = 2\sqrt{ax}$

$\frac{x^4}{6a^2} = 4ax$

$x^4 = 64a^3x$

$x(x^3 - 64a^3) = 0$

$\boxed{x=0}$, $x^3 - 64a^3 = 0$

$x^3 = 64a^3$

$\boxed{x = 4a}$

$y = \frac{x^2}{4a} \rightarrow x=0, y=0$

$x = 4a, \quad y = 4a$

$y = 2\sqrt{ax} \rightarrow x=0, y=0$

$x = 4a, y = 4a$

$(0,0)$ & $(4a, 4a)$

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{x=2\sqrt{ay}} xy \, dx \, dy$$

$$\int_0^{4a}\left[\frac{x^2}{2}y\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^{4a} y\left[4ay - \frac{y^4}{16a^2}\right] dy$$

$$= \frac{1}{2}\int_0^{4a} 4ay^2 - \frac{y^5}{16a^2}\,dy$$

$64\,a^{4/3}$

$$= \frac{1}{2}\left[4a\,\frac{y^3}{3} - \frac{1}{16a^2}\,\frac{y^6}{6}\right]_0^{4a}$$

$$= \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{y^6}{96a^2}\right]_0^{4a}$$

$$\frac{1}{2}\left[\frac{4a(4a)^3}{3} - \frac{(4a)^6}{96a^2}\right]$$

$$\frac{1}{2}\left[\frac{256a^4}{3} - \frac{4096a^4}{96\cancel{a^2}}\right]$$

$$\frac{1}{2}\left[\frac{256a^4}{3} - \frac{128a^4}{3\cancel{a^2}}\right]$$

$$\frac{1}{\cancel{2}}\left[\frac{\overset{64}{\cancel{128}}a^4}{3}\right]$$

$$= \frac{64}{3}a^4$$

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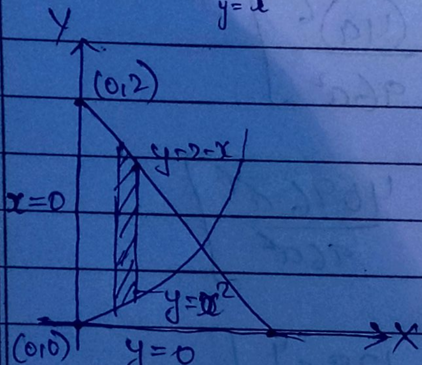
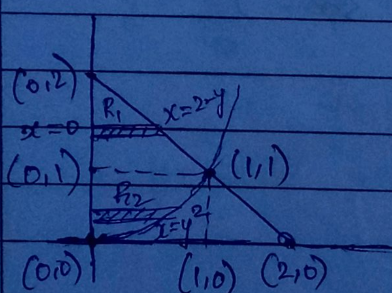
Evaluate

$\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2+y^2=a^2$

Evaluate $\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\, dy\, dx$ by changing order of integration



$x=0,\ x=1$ (on x-axis)
$y=x^2,\ y=2-x$ (on strip)



$y=x^2,\quad y=2-x$
$x^2=2-x$
$x^2+x-2=0$
$x=1,\quad x=-2$
$y=1,\ y=4$

$y=0$, to $y=1$ (on Y axis)
$x=0$ to $x=y^2$ (on strip)
$y=1$ to $y=2$ (on Y-axis)
$x=0$ to $x=2-y$ (on strip)

Intersection point
(1,1) (-2,4)

$$\int_{y=0}^{1}\int_{x=0}^{x=y^2} xy\, dx\, dy + \int_{y=1}^{y=2}\int_{x=0}^{x=2-y} xy\, dx\, dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]^{y^2} dy + \int_{y=1}^{2} y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y(y^4 - 0) + \frac{1}{2}\int_1^2 y\left[4-y^2\right]$$

$$= \frac{1}{2}\int_0^1 y^5\, dy + \frac{1}{2}\int_1^2 4y - y^3\, dy$$

$$= \frac{1}{2}\int_0^1 y^5\, dy + \frac{1}{2}\int_1^2 4y - y^3\, dy$$

$$= \frac{1}{2}\left[\frac{y^6}{6}\right]_0^1 + \frac{1}{2}\left[4 \times \frac{y^2}{\cancel{2}} - \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{y^6}{6}\right]_0^1 + \frac{1}{2}\left[2y^2 - \frac{y^4}{4}\right]_1^2$$

(#) Change the order of integration and hence evaluate

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} y^2 \cdot dx \cdot dy$$

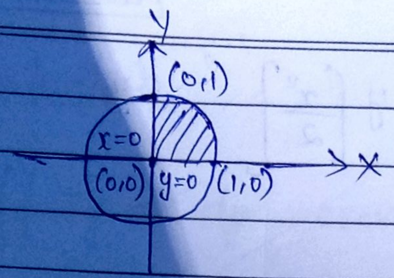
$y = \sqrt{1-x^2}$

$y =$

$y^2 = \sqrt{1-x^2}$ sq. on b.s

$y^2 = 1-x^2$

$x^2+y^2 = 1$



$x=0,\quad x=1$

$\sqrt{1-x^2}=0$

$$I=\int_{y=0}^{1}\int_{x=0}^{x=\sqrt{1-y^2}} y^2\,dx\,dy$$

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$$\int_{y=0}^{1}\left[y^2x\right]_0^{\sqrt{1-y^2}}dy$$

$$\int_{y=0}^{1}\left(y^2\sqrt{1-y^2}\right)dy$$

$x^2+y^2=1$

$y=\cos\theta$

$dy=-\sin\theta\,d\theta$

$0=\cos\theta$

$\theta=\pi/2$

$1=\cos\theta$

$\theta=\pi$

$$\int_{\theta=\pi/2}^{\pi}\cos^2\theta\left(\sqrt{1-\cos^2\theta}\right)\left(-\sin\theta\,d\theta\right)$$

$$-\int_{\pi/2}^{\pi}\cos^2\theta\,\sin^2\theta\,d\theta$$

$$\int \sin^m x\cos^n x\,dx$$

$$\frac{\left[(m-1)(m-3)\cdots\right]\left[(m-1)(m-3)\cdots\right]}{(m+n)(m+n-2)\cdots}$$

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$$= \left[\frac{(2-1)\,(2-1)}{(2+2)\,(2+2-2)}\cdot \pi/2\right.$$

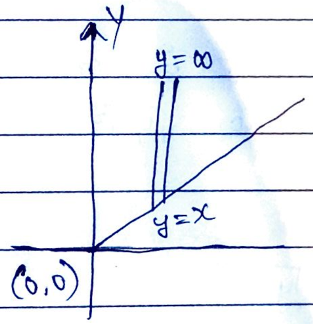
$$= -\frac{1}{8}\,\frac{\pi}{2} = -\frac{\pi}{16}$$

#

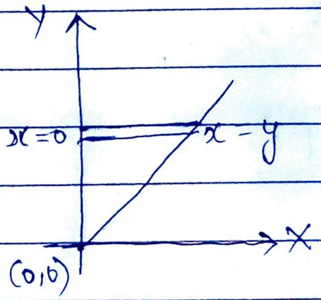
$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$$

Change the order of integration & Evaluate

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$$



$x = 0$, $x = \infty$ (on x-axis)
$y = x$, $y = \infty$ (on strip)



$y = 0$ to $y = \infty$
$x = 0$ to $x = y$

$$\int_{y=0}^{\infty} \int_{x=0}^{x=y} \frac{e^{-y}}{y}\,dx\,dy$$

$$\int_0^\infty \left[\frac{e^{-y}}{y}\,x\right]_0^y dy$$

$$\int_0^\infty \frac{e^{-y}}{\cancel{y}}\,(\cancel{y})\,dy$$

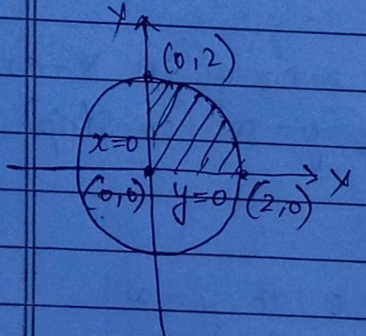
$$\int_0^\infty e^{-y}\,dy$$

$$-e^{-y}\Big]_0^\infty$$

$$\left[-e^{-\infty} - e^{0}\right] = 0 - 1 = \underline{\underline{-1}}$$

Evaluate $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (2-x)\,dy\,dx$

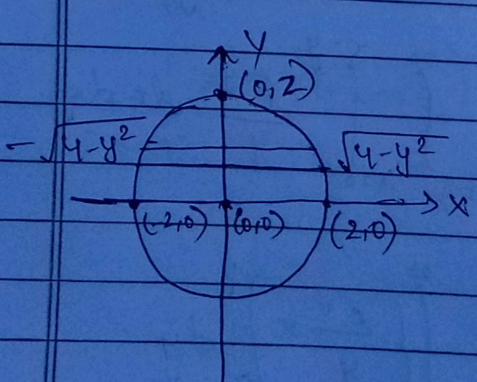
$$x = \int_{-2}^{2}\int_{y=0}^{\sqrt{4-x^2}}$$



$x = -2$ to 2 (on x-axis)

$y = 0$ to $\sqrt{4-x^2}$ (on strip)

$$\int_{0}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2-x)\,dy\,dx$$



$$\int_0^2 \left[2x - \frac{x^2}{2}\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}$$

$$\int_0^2 \left[2\sqrt{4-y^2} - \frac{4-y^2}{2}\right] - \left[2\sqrt{4-y^2} - \frac{4-y^2}{2}\right] dy$$

$$= \int_0^2 \left[2\sqrt{4-y^2} - \cancel{\frac{4-y^2}{2}} + 2\sqrt{4-y^2} + \cancel{\frac{4-y^2}{2}}\right] dy$$

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$$= \int_0^2 \left[4\sqrt{4-y^2} - \cancel{x}\,\frac{(4-y^2)}{\cancel{2}} \right] dy$$

$$= 2x - \frac{x^2}{2} \Big]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}$$

$$= 2\sqrt{4-y^2}$$

$$= 2\left[\sqrt{4-y^2} + \sqrt{4-y^2}\right] - \frac{1}{2}\left[\cancel{(4-y^2)} - \cancel{(4-y^2)}\right]$$

$$= 2 \cdot 2 \int_0^2 \sqrt{4-y^2}\, dy$$

$$4\left[\frac{y\sqrt{4-y^2}}{2} + \right. \qquad \Rightarrow 4\left[0 + 2\pi/2\right] = 4\pi$$

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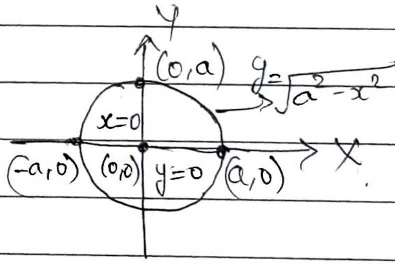
Change the integral

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\,dx$$ into polar form and hence evaluate



$x = r\cos\theta, \quad y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

r varies from 0 to a

θ varies from 0 to π

$x = -a, \quad y = 0$, $\quad x = a, \; Y = 0$

$(-a)^2 + 0 = r^2 \qquad a^2 = r^2$

$a^2 = r^2 \Rightarrow \boxed{r=a} \qquad \boxed{r=a}$

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$$\int_{r=0}^{a}\int_{\theta=0}^{\pi} \sqrt{r^2}\; r\,dr\,d\theta$$

$$\int_{0}^{a}\int_{0}^{\pi} r^2\,dr\,d\theta$$

$$\int_{0}^{\pi}\left[\frac{r^3}{3}\right]_0^a d\theta$$

$$\frac{1}{3}\int_{0}^{\theta}\left[a^3\,d\theta\right.$$

$$=\frac{1}{3}\left[a^3\theta\right]_0^{\pi}$$

$$\boxed{I=\frac{\pi a^3}{3}}$$

$$\int_{0}^{a}\int_{0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\,dx\,dy$$

$x = r\cos\theta,\quad y = r\sin\theta$

$x^2+y^2 = r^2$

$dy\,dx = r\,dr\,d\theta$

r varies from 0 to a

θ varies from 0 to $\pi/2$

$$r=\int_{0}^{a}\int_{0}^{\pi/2} r\sin\theta\cdot\sqrt{r^2}\; r\,dr\,d\theta$$

$$=\int_{0}^{a}\int_{0}^{\pi/2} r^3\sin\theta\,dr\cdot d\theta$$

$$= \int_0^a r^3 \left[-\cos\theta\right]_0^{\pi/2} dr$$

$$= -\int_0^a r^3 [0-1]\, dr$$

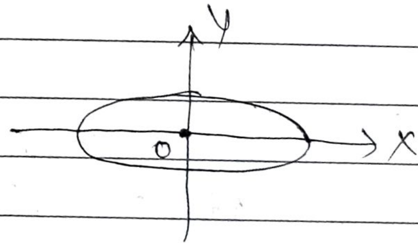
$$= -\int_0^a -r^3 \cancel{\theta}$$

$$= \int_0^a r^3\, dr$$

$$= \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}$$

Find the area of Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration

$$A = \int_{x=0}^{a} \int_{y=0}^{b}$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \frac{b}{a}\sqrt{a^2-x^2} \quad a / b/a \sqrt{a^2-x^2}$$

$$A = 4\int_{x=0}^{a} \int_{y=0} dy\, dx$$

$$= 4\int_0^a \left[y \right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$$

$$= \frac{4b}{a}\left[\left(0 + \frac{a^2}{2}\sin^{-1}(1)\right) \right]$$

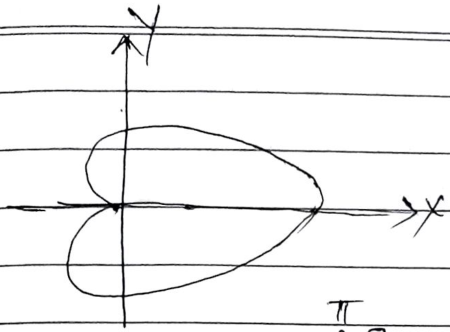
$$= \frac{\cancel{4}ab}{\cancel{2}} \cdot \frac{\pi}{\cancel{2}} = \pi ab$$

(#) Find the volume generated by the revolution of Cardiod $r = a(1+\cos\theta)$ above the initial line.

$$r = a(1+\cos\theta)$$

$$V = \iint_A 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$\int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$



$$2\pi \int_0^{\pi} \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{2\pi}{3} \int_0^{\pi} \left[a^3(1+\cos\theta)^3\right] \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1+\cos\theta)^3 \sin\theta \, d\theta$$

$1+\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$\theta = 0, \quad 1+\cos 0 = t$

$1+1 = t \Rightarrow \boxed{t=2}$

$\theta = \pi, \quad 1+\cos\pi = t$

$\boxed{t=1}$

$$\frac{2\pi a^3}{3} \int_2^{1} t^3(-dt)$$

$$= -\frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_2^1$$

$$= -\frac{\cancel{2}\pi a^3}{\cancel{12}_6}\left[1-16\right]$$

$$-\frac{\pi a^3}{\cancel{6}_2} \times \cancel{15}^5$$

$$\boxed{I = \frac{5\pi a^3}{2}}$$

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\# Find the volume of tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

Sol:-
$$\frac{z}{c}=1-\frac{x}{a}-\frac{y}{b}$$

$$\boxed{z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)}$$

$$z=0 \Rightarrow \frac{x}{a}+\frac{y}{b}=1$$

$$\boxed{y=b\left(1-\frac{x}{a}\right)}$$

$$z=0,\ y=0,\ \frac{x}{a}=1$$

$$\boxed{x=a}$$

$$V=\iiint dz\,dy\,dx$$

$$V=\int_{x=0}^{a}\int_{y=0}^{b(1-x/a)}\int_{z=0}^{c(1-x/a-y/b)} dz\,dy\,dx$$

$$=\int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} z\Big]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\,dx$$

$$=\int_0^a\int_0^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx$$

$\frac{Y^3}{3}$

$$= c\int_0^a \left[y - \frac{x}{a}\, y - \frac{y^2}{2b}\right]_a^{b(1-x/a)} dx$$

$$= c\int_0^a b\left(1-\frac{x}{a}\right) - \frac{x}{a}\, b\left(1-\frac{x}{a}\right) - \frac{1}{2\cancel{b}}\, b^{\cancel{2}}\left(1-\frac{x}{a}\right)$$

$$= c\int_0^a b\left(1-\frac{x}{a}\right)\left[\left(1-\frac{x}{a}\right) - \frac{1}{2}\left(1-\frac{x}{a}\right)\right] dx$$

$$= cb\int_0^a \left(1-\frac{x}{a}\right)\left(1-\frac{x}{a}\right)\left(1-\frac{1}{2}\right) dx$$

$$= \frac{cb}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{cb}{2}\, \frac{\left[\frac{\left(1-\frac{x}{a}\right)^3}{3}\right]^a}{-1/a}\, dx$$

$$= \frac{cb}{\cancel{2}}\left[\frac{-a\left(1-\frac{x}{a}\right)^3}{3}\right]_0^a$$

$$= -\frac{abc}{6}\left[(1-1)^3 - (1-0)^3\right]$$

$$= -\frac{abc}{6}\left[0-1\right]$$

$$= \frac{abc}{6}$$

(#) Find the volume of cylinder bounded by

$$x^2 + y^2 = 4$$
$$y + z = 4 \quad , \quad z = 0$$

Sol :-

$$z = 0 \text{ to } z = 4 - y$$
$$y^2 = 4 - x^2, \quad y = \pm\sqrt{4 - x^2}$$

$$y = -\sqrt{4-x^2} \text{ to } y = \sqrt{4-x^2}$$
$$y = 0, \quad x^2 = 4, \quad x = \pm 2$$
$$x = -2 \text{ to } x = 2$$

$$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-y} dy\, dx\, dz$$

$$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-y} dy\, dz\, dx$$

$$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[z\right]_0^{4-y} dy\, dx$$

$$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-y\right] dy\, dx$$

$$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-y\right] dy\, dx$$

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$$I = \int_{-2}^{2} \left[4y - \frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[4\sqrt{4-x^2} - 4\left(-\sqrt{4-x^2}\right)\right] - \frac{1}{2}\left[4-x^2 - (2)\sqrt{4-x^2}\right) dx\right]$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\, dx$$

$$= 8\int_{-2}^{2} \sqrt{2^2-x^2}\, dx$$

$$= 8\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= 8\left[\frac{2\sqrt{\cancel{4-4}}}{2} + 2\sin^{-1}(1) - \frac{2\sqrt{\cancel{4-4}}}{0} + 2\sin^{-1}\left(\frac{-2}{2}\right)\right]$$

$$= 16\left[\sin^{-1}(1) + \sin^{-1}(1)\right]$$

$$= 16\left(2\sin^{-1}(1)\right)$$

$$= \underline{\underline{16\pi}}$$

Total mass

(M_A) is given by

$$M_A = \iint_A \rho \, dx \, dy$$

Centre of Gravity

$$\bar{x} = \frac{\iint_A x \rho \, dx \, dy}{M_A}$$

$$\bar{y} = \frac{\iint_A y \rho \, dx \, dy}{M_A}$$

(M_V) is given by

$$M_V = \iiint_V \rho \, dx \, dy \, dz$$

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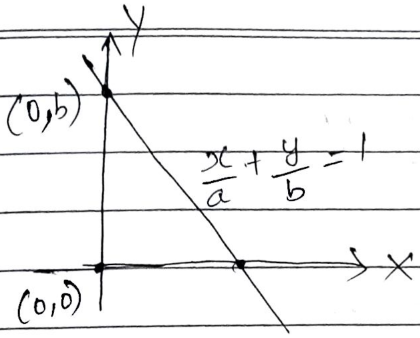
\# Find the centre of gravity of the triangular lamina bounded by the coordinate axis and the line $\frac{x}{a} + \frac{y}{b} = 1$

$$M_A = \iint_A \rho \, dx \, dy$$

$$\rho = xy.$$



$x \to 0$ to a

$y \to 0$ to $b\left(1-\frac{x}{a}\right)$

$$M_A = \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} xy \, dy \, dx$$

$$= \int_0^a \left[x \cdot \frac{y^2}{2}\right]_0^{b(1-x/a)} dx$$

$$= \frac{1}{2}\int_0^a x\, b^2\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{b^2}{2}\int_0^a x\left(1+\frac{x^2}{a^2}-2\frac{x}{a}\right) dx$$

$$= \frac{b^2}{2}\int_0^a \left(x+\frac{x^3}{a^2}-\frac{2x^2}{a}\right) dx$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2}+\frac{x^4}{4a^2}-\frac{2x^3}{3a}\right]_0^a$$

$$= \frac{b^2}{2}\left[\frac{a^2}{2}+\frac{a^{\cancel{4}2}}{4\cancel{a^2}}-\frac{2a^{\cancel{3}2}}{3\cancel{a}}\right]$$

$$= \frac{a^2b^2}{2}\left[\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right]$$

$$\boxed{M_A = \frac{a^2b^2}{24}}$$

$$\bar{x} = \frac{\iint_A x\,\rho\,dx\,dy}{M_A}$$

$$\iint_A x\,(xy)\,dx\,dy$$

$$\int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} x^2 y\,dy\,dx$$

$$\int_0^a \left[x^2 \frac{y^2}{2}\right]_0^{b(1-x/a)} dx$$

$$\frac{b^2}{2}\int_0^a \left[x^2\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)dx\right.$$

$$\frac{b^2}{2}\int_0^a \left(x^2+\frac{x^2}{a^2}-\frac{2x^3}{a}\right)dx$$

$$\frac{b^2}{2}\left[\frac{x^3}{3}+\frac{x^5}{5x^2}-\frac{2x^4}{4a}\right]_0^a$$

$$\frac{b^2}{2}\left[\frac{a^3}{3}+\frac{\cancel{a^5}\,a^3}{5\cancel{a^2}}-\frac{\cancel{2}\cancel{a^4}\,a^3}{{}_2\cancel{4a}}\right]$$

$$\frac{b^2a^3}{2}\left[\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right]$$

$$= \frac{a^3b^2}{60}$$

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$$\bar{x} = \frac{\frac{a^3 b^2}{\cancel{60}\,5}}{\frac{\cancel{a^2} \cancel{b^2}}{\cancel{24}\,2}}$$

$$\boxed{\bar{x} = \frac{2a}{5}}$$

$$\bar{y} = \frac{\iint_A y\,(dx\,dy}{M_A}$$

$$\iint_A y\,\rho\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} x\,y^2\,dy\,dx$$

$$\int_0^a \left[x\,\frac{y^3}{3}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$\frac{1}{3}\int_0^a x\,b^3\left(1-\frac{x}{a}\right)^3 dx$$

$$\frac{b^3}{3}\int_0^a x\left(1-\frac{x^3}{a^3}-\frac{3x}{a}+\frac{3x^2}{a^2}\right)dx$$

$$\frac{b^3}{3}\left[\frac{x^2}{2}-\frac{x^5}{5a^3}-\frac{3x^2}{3a}+\frac{3x^4}{4a^2}\right]$$

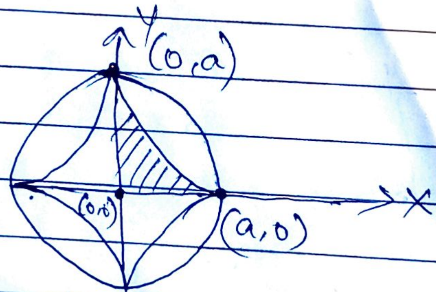
$$= \frac{a^2 b^3}{60}$$

$$\bar{y} = \frac{\frac{a^2 b^3}{\cancel{60}}}{\frac{a^2 b^2}{\cancel{24}}} = \frac{2b}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{2a}{5}, \frac{2b}{5}\right)$$

\# Find the centre of gravity of a lamina in the shape of Astroid $x^{2/3} + y^{2/3} = a^{2/3}$ represented in first quadrant.

Sol :-



$$x^{2/3} + y^{2/3} = a^{2/3}$$

$x \rightarrow 0$ to a

$$y^{2/3} = a^{2/3} - x^{2/3}$$

$$y = (a^{2/3} - x^{2/3})$$

$y \rightarrow 0$ to $y(x)$

$$M_A = \int_{x=0}^{a} \int_{y=0}^{y(x)} \rho \, dx \, dy$$

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$$\int_0^a \int_0^{y(x)} xy \, dy \, dx$$

$$\int_0^a \left[x \frac{y^2}{2} \right]_0^{y(x)} dx$$

$$= \frac{1}{2} \int_0^a x \left(\left(a^{2/3} - x^{2/3}\right)^{3/2} \right)^2 dx$$

$$= \frac{1}{2} \int_0^a x \left(a^{2/3} - x^{2/3} \right)^3 dx$$

$$= \frac{1}{2} \int_0^a x \left((a^{2/3})^3 - (x^{2/3})^3 - 3 (a^{2/3}) x^{2/3} + 3 (a^{2/3}) (x^{2/3})^2 dx \right.$$

$$= \frac{1}{2} \int_0^a x \left(a^2 - x^2 - 3a^{4/3} \cdot x^{2/3} + 3a^{2/3} x^{2/3} \right) dx$$

$$= \frac{1}{2} \int_0^a \left(xa^2 - x^3 - 3a^{4/3} x^{5/3} + 3a^{2/3} x^{2/3} \right) dx$$

$$= \frac{1}{2} \left[a^2 \frac{x^2}{2} - \frac{x^4}{4} - 3a^{4/3} \frac{x^{8/3}}{8/3} + 3a^{2/3} \frac{x^{a/3}}{a/3} \right]_0^a$$

$$= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} - 3 \times \frac{3}{8} a^{4/3} a^{8/3} + 3 \times \frac{3}{10} a^{2/3} a^{10/3} \right]$$

$$= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} - \frac{9}{8} a^4 + \frac{9}{10} a^4 \right]$$

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$$M_A = \frac{a^4}{80}$$

$$\bar{x} = \frac{\iint_A x \rho \, dx \, dy}{M_A}$$

$$\iint_A x (xy) \, dx \, dy$$

$$x = \int_0^a \int_0^{y(x)} x^2 y \, dx \, dy$$

$$= \int_0^a x^2 \left[\frac{y^2}{2}\right]_0^{y(x)} dx$$

$$= \int_0^a x^2 \left(a^{2/3} - x^{2/3}\right)^3 dx$$

$$= \int_0^a x^2 \left(a^2 - x^2 - 3a^{4/3} x^{2/3} + 3a^{2/3} x^{4/3}\right)$$

$$= \int_0^a \left[a^2 x^2 - x^4 - 3a^{4/3} x^{8/3} + 3a^{2/3} x^{4/3}\right] dx$$

$$= \frac{1}{2} \left[\frac{a^2 x^3}{3} - \frac{x^5}{3} - \frac{3a^{4/3} x^{11/3}}{11/3} + 3a^{2/3} \frac{x^{13/3}}{13/3}\right]_0^a$$

$$= \frac{1}{2} \left[\frac{a^2 a^3}{3} - \frac{a^5}{5} - \frac{9a^{4/3} a^{11/3}}{11} + \frac{a a^{2/3} a^{13/3}}{13}\right]$$

$= \frac{8a^5}{2145}$ $\quad \bar{x} = \frac{8a^3/2145}{a^4 180}$ $\quad \bar{x} = \frac{128a}{429}$

Similarly $\quad \bar{y} = \frac{128b}{429}$

$$(\bar{x}, \bar{y}) = \left[\frac{128a}{429}, \frac{128b}{429}\right]$$

V.V ***

β, Γ functions

(Γ = Gamma)

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad (m,n>0)$$

$$\Gamma(n) = \int_0^{x} e^{-x}\, x^{n-1}\,dx, \quad (n>0)$$

Properties

① $\beta(m,n) = \beta(n,m)$

② $\Gamma(n+1) = n\Gamma(n)$

③ $\Gamma(n) = (n-1)\,\Gamma(n-1)$

④ $\Gamma(n+1) = n!$

① $\beta(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \cdot d\theta$

② $\Gamma(n) = 2\int_{0}^{\infty} e^{-x^2} x^{2n-1}\, dx$

③ $\Gamma(m) = 2\int_{0}^{\infty} e^{-y^2} \cdot y^{2m-1}\, dy$

④ $\Gamma(m+n) = 2\int_{0}^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr$

Relationship b/w β and Γ functions

$$\beta(m,n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$$

$$\beta(m,n) = 2\int_{0}^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \cdot d\theta$$

$$\Gamma(m) = 2\int_{0}^{\infty} e^{-y^2} y^{2m-1}\, dy$$

$$\Gamma(n) = 2\int_{0}^{\infty} e^{-x^2} x^{2n-1}\, dx$$

$$\Gamma(m+n) = 2\int_{0}^{\infty} e^{-x^2} r^{2(m+n)-1} \cdot dr$$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2-y^2)}\, x^{2n-1}\, y^{2m-1}\, dx\, dy$$

put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy$

$x^2+y^2 = r^2$, $r \to 0$ to ∞, $\theta \to 0$ to $\pi/2$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\pi}\int_0^{\pi/2} e^{-x^2}(r\cos\theta)^{2n-1}(\sin\theta)^{2m-1}\, r\,dr.d\theta$$

$$= 4\int_0^{\infty}\int_0^{\pi/2} e^{-x^2}\, r^{2n-1}\, r^{2m-1}\, r\, \cos^{2n-1}\theta\, \sin^{2m-1}\theta .dr.d\theta$$

$$= 4\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\, r^{2(m+n)-1}\, \cos^{2n-1}\theta\, \sin^{2m-1}\theta\, dr.d\theta$$

$$= \left[2\int_{r=0}^{\infty} e^{-r^2}\, r^{2(m+n)-1}\, dr\right]\left[2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta . \cos^{2n-1}\theta . d\theta\right]$$

$$= \Gamma(m+n) . \beta(m,n)$$

$$\Gamma(m)\,\Gamma(n) = \Gamma(m+n)\,\beta(m,n)$$

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

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S.T $\Gamma(1/2) = \sqrt{\pi}$

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$m = n = 1/2$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(1/2) + (1/2)}{\Gamma(1)} \qquad \because \Gamma(1) = 1$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \left(\Gamma(1/2)\right)^2$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2\,\theta\Big]_0^{\pi/2}$$

$$= \cancel{2}\,\frac{\pi}{\cancel{2}} = \pi$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \left(\Gamma(1/2)\right)^2$$

$$\pi = \left(\Gamma(1/2)\right)^2$$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

$\boxed{\text{OR}}$

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$

$\Gamma(1/2) = 2\int_0^\infty e^{-x^2} x^0\, dx$

$\Gamma(1/2) = 2\int_0^\infty e^{-x^2}\, dx$

$\Gamma(m) = 2\int_0^\infty e^{-y^2} y^{2m-1}\, dy$

$\Gamma(1/2) = 2\int_0^\infty e^{-y^2}\, dy$

$(\Gamma(1/2))^2 = 4\int_0^\infty \int_0^y e^{-(x^2+y^2)}\, dx\, dy$

put $x = \cos\theta$, $y = r\sin\theta$

$dx\, dy = r\, dr\, d\theta$, $x^2+y^2 = r^2$

$r \rightarrow 0$ to ∞

$\theta \rightarrow 0$ to $\pi/2$

$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta$

put $r^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

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$t \to 0$ to ∞

$$= 4 \int_0^{\pi/2} \int_0^{\infty} e^{-t} \frac{dt}{2} \, d\theta$$

$$= \cancel{\frac{4}{2}}^{2} \int_0^{\pi/2} \left[-e^{-t}\right]_0^{\infty} d\theta$$

$$= 2 \int_0^{\pi/2} -[0-1] \, d\theta$$

$$= 2 \int_0^{\pi/2} d\theta = 2 \left[\theta\right]_0^{\pi/2}$$

$$= \frac{\cancel{2}\pi}{\cancel{2}} = \pi$$

$$\left(\Gamma(1/2)\right)^2 = \pi$$

$$* \quad \boxed{\Gamma(1/2) = \sqrt{\pi}}$$

Evaluate

$$\frac{\Gamma(3)\,\Gamma(2.5)}{\Gamma(5.5)}$$

$\Gamma(n) = (n-1)\,\Gamma(n-1)$

$\Gamma(n+1) = n\,\Gamma(n)$

$\Gamma(n+1) = n!$

$\Gamma(3) = \Gamma(2+1)$
$= 2!$

$\Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2}$

$\Gamma(2.5) = (2.5-1)\,\Gamma(2.5-1)$
$= 1.5\,\Gamma(1.5)$
$= 1.5\,(1.5-1)\,\Gamma(1.5-1)$
$= (1.5)(0.5)\,\Gamma(0.5)$

$0.75\ \cancel{2}\ \sqrt{\pi}$

$\Gamma(5.5) = (5.5-1)\,\Gamma(5.5-1)$
$= 4.5\,\Gamma(4.5)$
$= (4.5)(3.5)\,\Gamma(3.5)$
$= (4.5)(3.5)(2.5)\,\Gamma(2.5)$
$= (4.5)(3.5)(2.5)(1.5)\,\Gamma(1.5)$
$= (4.5)(3.5)(2.5)(1.5)(0.5)\,\Gamma(0.5)$

$$\frac{\Gamma(3)\,\Gamma(2.5)}{\Gamma(5.5)} = \frac{2!\ 2\cancel{\sqrt{\pi}}}{(4.5)(3.5)(2.5)(1.5)(0.5)\cancel{\sqrt{\pi}}}$$

$=$

Evaluate

$m = \frac{7}{2}, \quad n = -1/2$

$\beta\left(\frac{7}{2}, \frac{-1}{2}\right)$

$-\frac{1}{2} + \frac{1}{2}$

$\beta(m,n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

$= \frac{\Gamma(7/2) \cdot \Gamma(-1/2)}{\Gamma\left(\frac{7}{2} - \frac{1}{2}\right)}$

$= \frac{5/2 \cdot 3/2 \cdot 1/2 \; \Gamma(1/2) \; \frac{\Gamma(-1/2+1)}{-1/2}}{2!}$

$= \frac{\frac{15}{8} \; 1(1/2)\left(-\cancel{2}\,\Gamma(1/2)\right)}{\cancel{2}} = \frac{-15}{8}\pi$

Evaluate

$\int_0^\infty x^{3/2} \cdot e^{-x} \, dx$

$\frac{3}{2} \quad 2$

$I = \int_0^\infty x^{3/2} \cdot e^{-x} \cdot dx$

$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} \, dx$

$$n-1 = \frac{3}{2}$$

$$n = \frac{5}{2}$$

$$\Gamma(5/2) = 3/2 \cdot 1/2 \; \Gamma(1/2)$$

$$= \frac{3}{4}\sqrt{\pi}$$

\# $$\int_0^\infty \sqrt{y}\, e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy$$

$$I_1 = \int_0^\infty y^{1/2} e^{-y^2} dy$$

$$I_2 = \int_0^\infty e^{-y^2} y^{-1/2} dy$$

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$$

$I_1 \Rightarrow \; 2n-1 = 1/2$

$n = 3/4$

$\Gamma(3/4)$

$I_2 = 2n-1 = -1/2$

$n = 1/4$

$\Gamma(1/4)$

$$I_1 \times I_2 = \Gamma(3/4)\,\Gamma(1/4)$$

$$= \pi\sqrt{2}$$

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**

(#) S.T $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}\, d\theta \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta$$

$$p = 2m-1, \quad q = 2n-1$$

$$m = \frac{p+1}{2}, \quad n = \frac{q+1}{2}$$

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta$$

$$\boxed{\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta}$$

$$I_1 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^0\theta \cdot d\theta$$

$$I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^0\theta \cdot d\theta$$

$$I_1 = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right)$$

$$I_1 = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{1}{2}\left[\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}\right]$$

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$$I_2 = \frac{1}{2} \beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)}$$

$$I_1 \times I_2 = \frac{1}{2} \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(\cancel{3/4})} \times \frac{1}{2} \frac{\Gamma(\cancel{3/4})\,\Gamma(1/2)}{\Gamma(5/4)}$$

$$= \frac{\pi}{4} \frac{\Gamma(1/4)}{\Gamma(5/4)}$$

$$= \frac{\pi}{4} \frac{\Gamma\cancel{(1/4)}}{1/4\,\Gamma\cancel{(1/4)}}$$

$$= \frac{\pi}{\cancel{4}} \times \cancel{4} = \pi$$

(4) $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\, d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta \cdot d\theta$$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta$$

$$= \frac{1}{2} \beta\left(\frac{1/2+1}{2}, \frac{-1/2+1}{2}\right)$$

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$$= \frac{1}{2} \beta \left(\frac{3}{4}, \frac{1}{4}\right)$$

$$\frac{\frac{1}{2} \Gamma(3/4)\, \Gamma(1/4)}{\Gamma(1)}$$

$$\frac{1}{\cancel{2}} \times \frac{\cancel{2}\,\pi\sqrt{2}}{1} = \frac{\pi}{\sqrt{2}}$$

$$I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\, d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{1/2}\theta\, d\theta$$

$$= \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \cdot d\theta$$

$$= \frac{1}{2} \beta\left(\frac{1/2+1}{2}, \frac{-1/2+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(3/4, \frac{1}{4}\right)$$

$$= \frac{\frac{1}{2} \Gamma(3/4)\, \Gamma(1/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \times \frac{\pi\sqrt{2}}{1} = \pi/\sqrt{2}$$

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Evaluate $\int_0^2 (4-x^2)^{3/2}\, dx$ in terms of β functions

and hence evaluate the integral.

$$I = \int_0^2 (4-x^2)^{3/2}\, dx$$

$(a^n - x^n) \Rightarrow x^n = a\sin^2\theta$

$(a^2 + x^n) \Rightarrow x^n = a\tan^2\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$x = 0 \Rightarrow 0 = 2\sin\theta$

$\boxed{\theta = 0}$

$x = 2 \Rightarrow \cancel{2} = \cancel{2}\sin\theta$

$\boxed{\theta = \pi/2}$

$(4-x^2)^{3/2} = (4-4\sin^2\theta)^{3/2}$

$(4(1-\sin^2\theta))^{3/2}$

$(2^2(\cos^2\theta))^{3/2}$

$2^3\cos^3\theta = 8\cos^3\theta$

$$I = \int_0^{\pi/2} 8\cos^3\theta \cdot 2\cos\theta\, d\theta$$

$$= 16\int_0^{\pi/2} \cos^4\theta \cdot d\theta$$

$$= 16\int_0^{\pi/2} \sin^0\theta\, \cos^4\theta\, d\theta$$

$$= \overset{8}{\cancel{16}} \times \frac{1}{\cancel{2}} \beta\left(\frac{0+1}{2}, \frac{4+1}{2}\right)$$

$$8\beta\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$\frac{8\,\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\frac{8\Gamma(1/2)\,\Gamma(5/2)}{\Gamma(3)}$$

$$\frac{\overset{\cancel{2}}{\cancel{4}}\,\cancel{8}\sqrt{\pi}\;3/\cancel{2}\cdot 1/\cancel{2}\;\Gamma(1/2)}{\cancel{2}!}$$

$$= 3\pi$$

Evaluate $\int_0^\infty \frac{dx}{1+x^4}$ in terms of β

$$I = \int_0^\infty \frac{dx}{1+x^4}$$

$x^4 = \tan^2\theta$

$x = \tan^{1/2}\theta$

$dx = \tan^{-1/2}\theta.\quad \sec^2\theta \; \frac{1}{2}\, d\theta$

$x = 0 \Rightarrow 0 = \tan^{1/2}\theta$

$\boxed{\theta = 0}$

$x = \infty \Rightarrow \infty = \tan^{1/2}.\theta$

$\boxed{\theta = \pi/2}$

$$I = \int_0^{\pi/2} \frac{1/2 \tan^{-1/2}\theta \cdot \sec^2\theta \, d\theta}{1+\tan^2\theta}$$

$$\frac{1}{2}\int_0^{\pi/2} \frac{\tan^{-1/2}\theta \cdot \cancel{\sec^2\theta} \cdot d\theta}{\cancel{\sec^2\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^{-1/2}\theta}{\cos^{-1/2}\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^{1/2}\theta \cdot d\theta$$

$$\frac{1}{2}\left[\frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right)\right.$$

$$= \frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma(1/4)\,\Gamma(3/4)}{\Gamma(1)}$$

$$\frac{1}{4}\pi\sqrt{2} = \frac{\pi\sqrt{2}}{4}$$