Module 2

Chapter: 3 Elastic Properties of Materials.

Elasticity

It is that property of a ~~m~~ body due to which it regains its original shape and size when the deforming force is removed.

Stress

The restoring force per unit area developed inside the body is called stress. Stress is given by the ratio of the applied force to the area of its application.

Strain

The ratio of the change in diamension to original diamensions is called strain.

Tensile stress (Longitudinal stress)

It is the stretching force acting per unit area of the section of the solid along its length. If 'F' is the force applied normally to a cross sectional area 'a', then the stress is F/a

$$\text{Longitudinal stress} = \frac{F}{a}$$

Linear Strain (Longitudinal Strain)

If 'x' is the change in length produced in an original length 'L'. Then,

$$\frac{\text{Change in length}}{\text{Original length}} = \frac{x}{L}$$

$$\text{Linear strain or longitudinal strain} = \frac{x}{L}$$

Compressive stress or Volume stress

It is the uniform pressure ~~(force)~~ (force per unit area) acting normally all over the body.

$$\text{The compressive stress} = \frac{F}{a}$$

Volume strain

If 'v' is the change in volume in an original volume 'V' of the body, then,

$$\text{Volume strain} = \frac{v}{V}$$

Shear Stress or Tangential Stress

It is the force acting tangentially per unit area on the surface of a body.

$$\text{Tangential stress} = F/a$$

Shear strain

It is the ratio of change in dimension to original dimension.

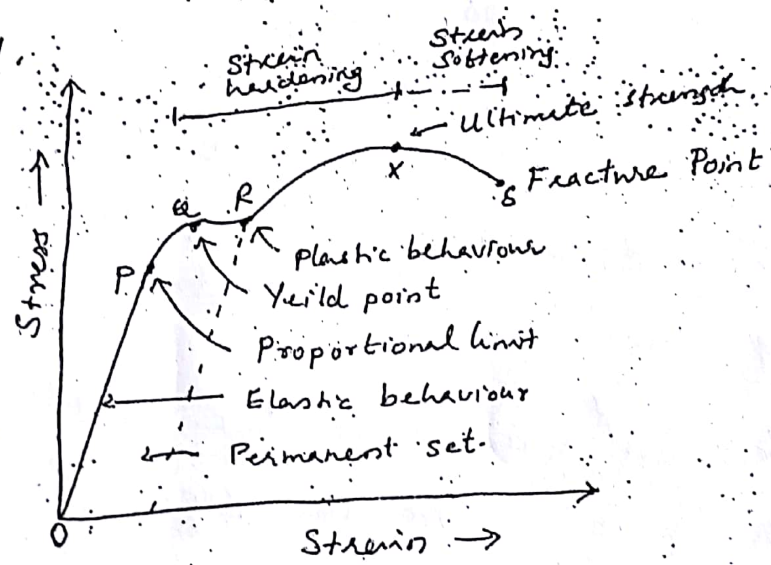
$$\text{Shearing angle} = \text{Shear strain } \theta = \frac{x}{L}$$

= The ~~rigidity~~ rigidity
material can be found

Hooke's law and Stress- Strain Diagrams

Stress is always proportional to strain but only when the deforming force is small. This is called Hooke's law.

For small deformation, the molecular displacement will not be too large so that the molecules can return to their previous positions once the external force is removed.



The stress versus strain plot for a metallic wire or a rod is shown in figure. where OP is the portion of linearity where stress is directly proportional to strain. But the body continuous to show perfect elastic property even up to the point Q. But the dependence between stress and strain is not observed in this portion. The points P and Q marks the

"Proportional limit" and the "Elastic limit" respectively. B is called the yield point. At any point between B and D the body fails to regain its original length if the deforming force is removed but traverses the dashed line in fig and the material therefore is said to be permanent set. Such a deformation is called "Plastic Deformation". At 'E' the deformation becomes large enough to pull the molecules far apart. E is called the 'fracture point'. The force per unit area at which the wire breaks is called 'breaking stress'.

Between B and S is the plastic range. This is divided in to two regions: Strain hardening region and strain softening region. From B to K, K is the peak point on the curve. A kind of hardening called strain hardening takes place in the material with increase in stress. From K to S, the material weakens, and called strain softening. K is the limit up to which the material stay fit and is referred as ultimate strength. Based on this observation Hooke's law is stated as " the stress is directly proportional to the strain within the

ELASTIC MODULI

1 Young's Modulus (Y)

The ratio of the longitudinal stress to linear strain, within the elastic limit is called Young's modulus. It is denoted by the letter Y.

$$Y = \frac{\text{longitudinal stress}}{\text{linear strain}}$$

$$Y = \frac{F/a}{x/L}$$

$$Y = \frac{FL}{ax} \quad \text{—— (1)}$$

S I unit of Y is N/m^2.

2 Bulk Modulus (K)

The ratio of the compressive stress to the volume strain without change in shape of the body within the elastic limit is called the Bulk modulus. It is denoted by the letter k.

$$K = \frac{\text{compressive stress}}{\text{Volume strain}}$$

$$= \frac{F/a}{v/V}$$

Pressure $P = F/a$

$$K = \frac{P}{v/V}$$

$$K = \frac{PV}{} \quad \text{—— (2)}$$

3. Rigidity Modulus (n)

Rigidity modulus is defined as the ratio of the tangential stress to the shearing strain. It is denoted by the letter 'n'.

$$n = \frac{\text{tangential stress}}{\text{shearing strain}}$$

$$n = \frac{F/a}{x/L}$$

$$n = \frac{FL}{ax} \quad \text{--- (3)}$$

SI unit of 'n' is N/m^2.

Longitudinal strain coefficient 'α'

The longitudinal strain produced per unit stress is called longitudinal strain coefficient.

longitudinal strain $= \frac{x}{L}$

Let 'T' be the applied stress

α = longitudinal strain produced / unit stress

$$\alpha = \frac{x/L}{T} = \frac{x}{LT}$$

$$\alpha = \frac{x}{TL} \quad \text{--- (4)}$$

$$x = \alpha TL$$

Lateral deformation: The change which occurs in a direction perpendicular to the direction along which the deforming force is acting is called lateral change.

Perfectly elastic body

If a body under stress within its elastic limit, returns to the original form without any trace of deformation soon after the removal of the deforming force, is called a perfectly elastic body.

Plastic body -

If a body deformed under stress, retains its deformed shape and size even when the stress is decreased or removed, it is called a plastic body.

Importance of elasticity in Engineering applications

Consider the example of iron and steel. When a tool made of iron is used in an application where there is a lot of vibration, a small fracture is formed in the tool. When in use, the fracture propagates in the body of the tool and end up shattering it suddenly. In case the tool is made of steel, it springs back to shape repeatedly as steel is more elastic. Thus the engineers can make better choice of the material for their use by knowing the nature of its stress-strain curve.

Pure metals are soft by property. They are ductile. Hence they are rarely used for engineering applications. Alloys are harder than pure metals. They are produced by blending different metals. They exhibit unique properties that are different to the constituent metals and offer better elastic properties useful for engineering application.

Strain Hardening and Strain Softening

Certain materials that are plastically deformed earlier are stressed again, show an increased yield point. This effect is called strain hardening.

For certain materials the stress-strain curve will be as shown in fig. The curve will have negative slope soon after the elastic region. The stress is negative slope indicate that there are the softening effect of the material. This effect is called strain softening.

Lateral strain

If a deforming force produces a change 'd' in its diameter when the original diameter is D then,

$$\text{Lateral strain} = \frac{d}{D}$$

Lateral strain coefficient 'β'

The lateral strain produced per unit stress is called lateral strain coefficient.

$$\text{lateral strain} = \frac{d}{D}$$

$$\beta = \text{lateral strain produced / unit stress}$$

$$= \frac{d/D}{F}$$

$$\beta = \frac{d}{FD} \quad \text{—— (5)}$$

Poisson's Ratio (σ)

Within the elastic limit of a body, the ratio of lateral strain to the longitudinal strain is a constant and is called Poisson's Ratio. It is represented by the symbol 'σ'.

$$\sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$= \frac{d/D}{l/L}$$

$$\sigma = \frac{Ld}{lD} \quad \text{—— (6)}$$

There is no unit for Poisson's ratio.

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$$\beta = \text{Lateral Strain produced / unit stress}$$

$$= \frac{d/D}{T}$$

$$\beta = \frac{d}{TD} \quad \text{— (5)}$$

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$$\sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$= \frac{d/D}{x/L}$$

$$\sigma = \frac{Ld}{xD} \quad \text{— (6)}$$

There is no unit for poisson's ratio.

Consider the ratio $\frac{\beta}{\alpha}$.

$$\frac{\beta}{\alpha} = \frac{(d/TD)}{(x/TL)}$$

$$\frac{\beta}{\alpha} = \frac{Ld}{xD} \quad \text{--- (7)}$$

R.H.S of eqn (6) and (7) are same.

$$\therefore \sigma = \frac{\beta}{\alpha} \quad \text{--- (8)}$$

Expression for Y, K and n in terms of α and β

1. $Y = \frac{1}{\alpha}$

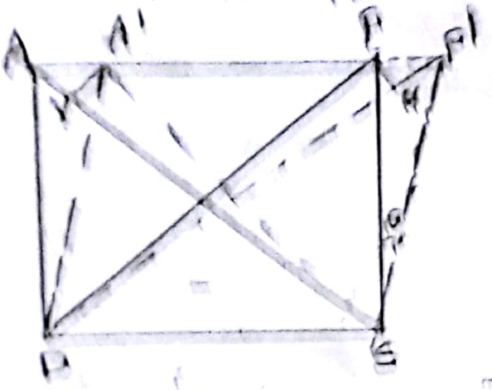
2. $K = \frac{1}{3(\alpha - 2\beta)}$

3. $n = \frac{1}{2(\alpha+\beta)}$

Relation between the elastic constants

Relation between Shearing strain, Elongation strain, and Compression strain

Consider a cube whose lower surface is fixed to a rigid support. Let APSD be one of its faces with the side DS along the fixed support (fig). when a deforming force is applied along the upper face towards AP, causes ~~a~~ displacements at different parts of the cube. A moves to A' and P to P'. The diagonal AS reduces to a length is and PD is stretched to a length P'D. If P'X is drawn perpendicular to PD and A'Y to AS, then DP = DX and A'S = YS



Elongation strain $= \frac{P'X}{PD}$ — (1)

Compression strain $= \frac{AY}{AS}$ — (2)

If 'L' is the length of each side of the cube, then, $AS = PD = \sqrt{2}L$ (By pythagoras theorem) — (3)

Now, $P'X = PP' \cos \angle PP'X$

$= PP' \cos \angle AP'D$

In the isosceles right angled triangle APD, $\angle APD = 45°$

Since 'θ' is very small, $\angle AP'D = \angle APD = 45°$

$P'X = PP' \cos 45° = \frac{PP'}{\sqrt{2}}$ — (4)

Substituting from eqn (3) and (4) eqn (1) becomes

Elongation strain $= \frac{PP'}{2L} = \frac{\theta}{2}$ — (5)

$[\because \theta = \frac{PP'}{L}]$

IIIly Compression strain along AS $= \frac{\theta}{2}$ — (6)

Adding (5) and (6)

... strain $= \frac{\theta}{2} + \frac{\theta}{2} = \theta$

Relation between Y, n and σ

Consider a cube with each of its sides of length L. Let APSD be one of its faces with the side DS along the fixed support (fig). When a tangential force F is applied at the upper face, it turns the faces through an angle θ. So ~~A become~~ A' and P changes to P'. Also DP increases to DP'.

If α and β are the longitudinal and lateral strain coefficients along DP per unit stress, then, Extension produced for the length DP due to tensile stress $= T \cdot DP \cdot \alpha$ and Extension produced for the length DP due to compression stress $= T \cdot DP \cdot \beta$.

Total extension along $DP = DP \cdot T \cdot (\alpha+\beta)$

From (fig) Total extension produced $= P'X$.

$$P'X = DP \cdot T\,(\alpha+\beta)$$

$$P'X = (\sqrt{2}L)\,T\,(\alpha+\beta)$$

From eqn 4, $P'X = \frac{PP'}{\sqrt{2}} = \frac{x}{\sqrt{2}}$ $\quad (\because PP' = x)$.

$$\sqrt{2}L\,T(\alpha+\beta) = \frac{x}{\sqrt{2}}$$

$$2(\alpha+\beta) = \frac{x}{TL}$$

Inverting, $\frac{1}{2(\alpha+\beta)} = \frac{TL}{x} = \frac{T}{x/L} = \frac{T}{\theta} = n.$

$$n = \frac{1}{2(\alpha+\beta)}$$

$$= \frac{1}{2\alpha(1+\beta/\alpha)}$$

$$n = \frac{1/\alpha}{2(1+\sigma)} \quad (\because \sigma = \beta/\alpha) \quad \text{--- (7)}$$

Young's Modulus,

$$Y = \frac{\text{Stress}}{\text{longitudinal Strain}} = \frac{1}{\text{long. strain}/\text{stress}}$$

$$Y = \frac{1}{\text{Strain along DP}/\text{unit stress}} = \frac{1}{\alpha}$$

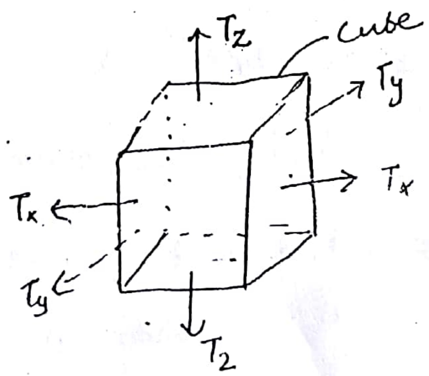
Eqn. (7) becomes,

$$n = \frac{Y}{2(1+\sigma)}$$

$$Y = 2n(1+\sigma) \quad \text{--- (8)}$$

Relation between K, Y and σ

Consider a cube of unit length, breadth and height. Let T_x, T_y and T_z be the outward stresses acting along the x, y and z directions as indicated in fig. Let 'α' be the elongation per unit length per unit stress along the directions of the forces and 'β' be the contraction per unit length per unit stress in a direction $\perp$ar to the respective forces. Then a stress T_x produces an increase in



length of αT_x to x direction and contraction βT_y and βT_z respectively in the cube along x direction. Hence a length which was unity along x direction becomes,

$$1+\alpha T_x - \beta T_y - \beta T_z$$

Similarly along y and z directions, the respective lengths become,

$$1+\alpha T_y - \beta T_x - \beta T_z \text{ and}$$

$$1+\alpha T_z - \beta T_y - \beta T_x$$

Hence the new volume of the cube is

$$= (1+\alpha T_x - \beta T_y - \beta T_z)(1+\alpha T_y - \beta T_x - \beta T_z)$$
$$(1+\alpha T_z - \beta T_x - \beta T_y)$$

Since α and β are very small, the volume of the cube $= 1+\alpha(T_x+T_y+T_z) - 2\beta(T_x+T_y+T_z)$

$$= 1+(\alpha-2\beta)(T_x+T_y+T_z)$$

If $T_x = T_y = T_z$,

then the volume $= 1+(\alpha-2\beta)3T$

Since the cube under consideration is of unit volume, increase in volume $= [1+3T(\alpha-2\beta)]-1$

$$= 3T(\alpha-2\beta)$$

If instead of outward stress T, a pressure P is applied the decrease in volume $= 3P(\alpha-2\beta)$

Volume strain = $\frac{\text{change in volume}}{\text{original volume}}$

$$= \frac{3P(\alpha - 2\beta)}{1}$$

Bulk modulus K = $\frac{\text{Pressure}}{\text{Volume strain}}$

$$= \frac{P}{3P(\alpha - 2\beta)}$$

$$= \frac{1}{3(\alpha - 2\beta)}$$

$$= \frac{1}{3\alpha(1 - 2\beta/\alpha)}$$

$$= \frac{\frac{1}{\alpha}}{3(1 - 2\sigma)} \quad (\because \sigma = \beta/\alpha)$$

$$= \frac{Y}{3(1 - 2\sigma)}$$

$$K = \frac{Y}{3(1 - 2\sigma)} \quad \text{———} \quad (9)$$

Relation between K, n and Y:

The relation between Y, n and σ is given by

$Y = 2n(1+\sigma)$. as per eqn (8)

and for K, Y and σ as,

$K = \frac{Y}{3(1-2\sigma)}$ as per eqn (9).

$Y = 3K(1-2\sigma)$

From eqn (8), $\frac{Y}{n} = 2 + 2\sigma$

" eqn (9), $\frac{Y}{3K} = 1 - 2\sigma$

Adding the above eqns,

$$\frac{Y}{n} + \frac{Y}{3k} = 3$$

$$Y\left[\frac{1}{n} + \frac{1}{3k}\right] = 3$$

$$Y\left[\frac{3k+n}{3nk}\right] = 3$$

$$Y = \left[\frac{9nk}{3k+n}\right] \quad \text{--- (10)}$$

Relation between K, n and σ

We have,

$$Y = 2n(1+\sigma) \quad \text{--- (11)}$$

$$Y = 3k(1-2\sigma) \quad \text{--- (12)}$$

Adding from eqns (11) and (12), we get,

$$2n + 2n\sigma = 3k - 6k\sigma$$

$$2n\sigma + 6k\sigma = 3k - 2n$$

$$\sigma(2n+6k) = 3k - 2n$$

$$\sigma = \frac{3k-2n}{2n+6k} \quad \text{--- (13)}$$

Limiting values of σ

From eqns (8) and (9), we have,

$$Y = 2n(1+\sigma) \text{ and } Y = 3k(1-2\sigma)$$

$$2n(1+\sigma) = 3k(1-2\sigma) \quad \text{--- (14)}$$

From these relations, we can see that,
the values of σ always lie between -1 and 0.5

In practice, σ is usually taken between 0 and 0.5

BENDING OF BEAMS

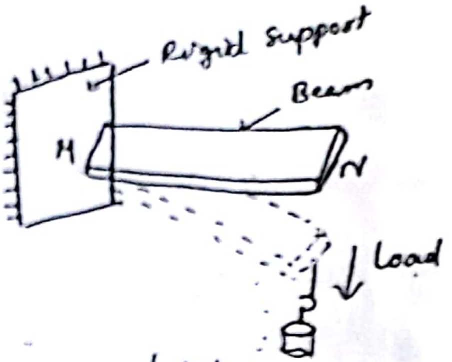
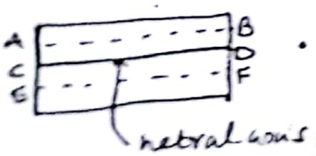


fig 1

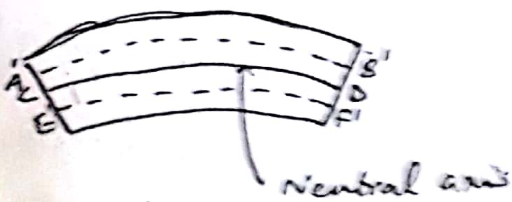
A homogeneous body of uniform cross section whose length is large compared to its other dimensions is called a beam.

If an arrangement is made to fix one end of the beam to a rigid support and its other end loaded, the arrangement is called 'single cantilever' or simply the cantilever fig(1)

Neutral surface and neutral axis



fig(2)



fig(3)

If a load is attached to the free end of a beam, the beam bends. The beam can be made up of a number of parallel layers and each layer is made up of a number of filament. A filament like AB of an upper layer will be elongated to A'B' and the one like EF of a

lower layer will be contracted to EF'. But there will always be a particular layer whose filament do not change their length as shown for CD.
Such a layer is called neutral surface (neutral plane), and the line along which a filament of it is situated is called neutral axis.

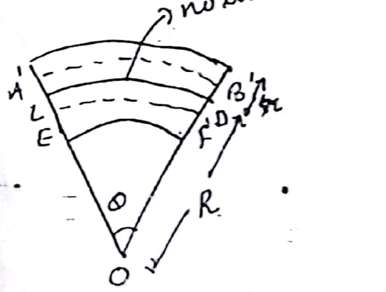
Definitions

Neutral surface is that layer of a uniform beam which does not undergo any change in its diamensions when the beam is subjected to bending within its elastic limit.

Neutral axis is a longitudinal line along which the neutral surface is intercepted by any longitudinal plane considered in the plane of bending.

Bending moment of a beam

Consider a long unifrom beam whose one end is fixed. The beam can be made up of a number of parallel layers. The shape of different layers of the bend beam can be imagined to form part of concentric circles of varying radii as shown in fig. Let 'R' be the radius of the circle



part of bend beam

in which the neutral surface forms a part.

$$EF = R\theta$$

where 'θ' is the common angle subtended by the layers at the common centre 'O' of the circles. Now the layer AB has been elongated to A'B'.

$$\text{Change in length} = A'B' - AB$$

$$AB = EF = R\theta$$

If the successive layers are separated by a distance 'r' then,

$$A'B' = (R+r)\theta$$

$$\text{Change in length} = (R+r)\theta - R\theta = r\theta$$

But original length $= AB = R\theta$.

$$\text{Linear strain} = \frac{r\theta}{R\theta} = \frac{r}{R} \quad \text{— (1)}$$

$$\text{Longitudinal stress} = Y \times \text{linear strain}$$
$$= Y\left(\frac{r}{R}\right) \quad \text{— (2)}$$

But stress $= F/a$.

where, F, is the force acting on the beam and 'a' is the area of the layer AB

$$\frac{F}{a} = Y\frac{r}{R}$$

$$F = \frac{Yar}{R}$$

Moment of this force about the neutral axis

$= F \times$ its distance from neutral axis

$$= F \times r = \frac{Yar^2}{R}$$

Moment of the force acting on the entire beam $= \sum \frac{Yar^2}{R}$.

$= \frac{Y}{R} \sum ar^2$. —— (4)..

The moment of inertia of a body about a given axis is given by $\sum mr^2$ where $\sum m$ is the mass of the body. Similarly $\sum ar^2$ is called the geometric moment of inertia I_g.

$I_g = \sum ar^2 = AK^2$ where A is the area of cross section of the beam and K is the radius of gyration about the neutral axis.

~~Moment of inertia of the~~

Moment of the force $= \frac{Y}{R} I_g$

or bending moment $= \frac{Y}{R} I_g$ —— (5)

Expression for bending moment

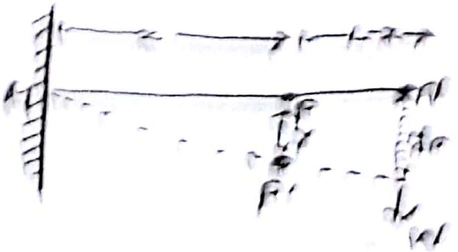
(1) Bending moment for a beam of rectangular cross-section $= \frac{Y}{R}\left[\frac{bd^3}{12}\right]$ —— (6)

where 'b' and 'd' are the breadth and thickness of the beam respectively.

(2) Bending moment for a beam of circular cross section $= \frac{\pi Y}{4R} r^4$

where 'r' is the radius of the beam.

Single Cantilever



Consider a uniform beam of length L fixed at M. Let a load 'W' act on the beam at N. As a result the beam bends as shown in fig.

Consider a point 'P' on the beam at a distance 'x' from the fixed end which will be at a distance of (L-x) from N. Let 'P'' be the position after the beam is bent.

∴ Bending moment = Force × perpendicular distance

$= W(L-x)$

But bending moment $= \frac{Y}{R} I_g$

$$\therefore \frac{Y}{R} I_g = W(L-x)$$

$$\frac{1}{R} = \frac{W(L-x)}{Y I_g} \quad \text{--- (1)}$$

If 'y' be the point P' then it can be shown that $\frac{1}{R} = \frac{d^2y}{dx^2}$ --- (2)

where R is the radius of the circle to which the bend beam becomes a part.

Comparing eqn (1) and (2)

$$\frac{d^2y}{dx^2} = \frac{W(L-x)}{YI_g}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{W(L-x)}{YI_g}$$

$$d\left(\frac{dy}{dx}\right) = \frac{W}{YI_g}(L\,dx - x\,dx)$$

Integrating, $$\frac{dy}{dx} = \frac{W}{YI_g}\left[Lx - \frac{x^2}{2}\right] + C_1 \quad \text{--- (3)}$$

where C_1 is the constant of integration.

But $\frac{dy}{dx} = 0$ at $x = 0$, then $C_1 = 0$ in eqn (3).

~~Then~~ Eqn (3) becomes, ~~$0 = C_1$~~

$$\frac{dy}{dx} = \frac{W}{YI_g}\left[Lx - \frac{x^2}{2}\right]$$

$$dy = \frac{W}{YI_g}\left[Lx - \frac{x^2}{2}\right]dx$$

Integrating both sides,

$$Y = \frac{W}{YI_g}\left[L\frac{x^2}{2} - \frac{x^3}{6}\right] + C_2 \quad \text{--- (4)}$$

where C_2 is the constant of integration.

But $y = 0$ at $x = 0$. i.e. $0 = C_2$

Substituting C_2 in eqn (4),

$$y = \frac{W}{YI_g}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right]$$

At the loaded end $y = y_0$ and $x = L$.

$$y_0 = \frac{W}{Y I_g}\left[\frac{L^3}{2} - \frac{L^3}{6}\right]$$

Depression produced at the loaded end is,

$$y_0 = \frac{WL^3}{3 Y I_g} \quad \text{——— (5)}$$

Young's Modulus, $$Y = \frac{WL^3}{3 y_0 I_g} \quad \text{——— (6)}$$

If the beam is having rectangular cross section, with breadth 'b' and thickness 'd'

I_g is given as

$$I_g = \frac{bd^3}{12}$$

Substituting in eqn (6),

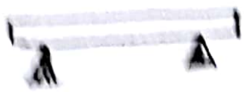
$$Y = \frac{WL^3}{3 y_0} \times \frac{12}{bd^3}$$

$$Y = \frac{4 WL^3}{y_0 bd^3} \quad \text{——— (7)}$$

Different types of beams and their engineering applications

There are four different types of beams.

1. Simple beam
2. Continuous beam
3. Cantilever beam
4. Fixed beam



Simple beam

A simple beam is a bar resting upon supports at each ends.

Continuous beam

A continuous beam is a ~~beam whose one end is fixed~~ or bar resting upon more than two supports.

Cantilever beam

A cantilever beam is a beam whose one end is fixed and the other end is free.

Fixed beam

A beam fixed at both ends is called a fixed beam.

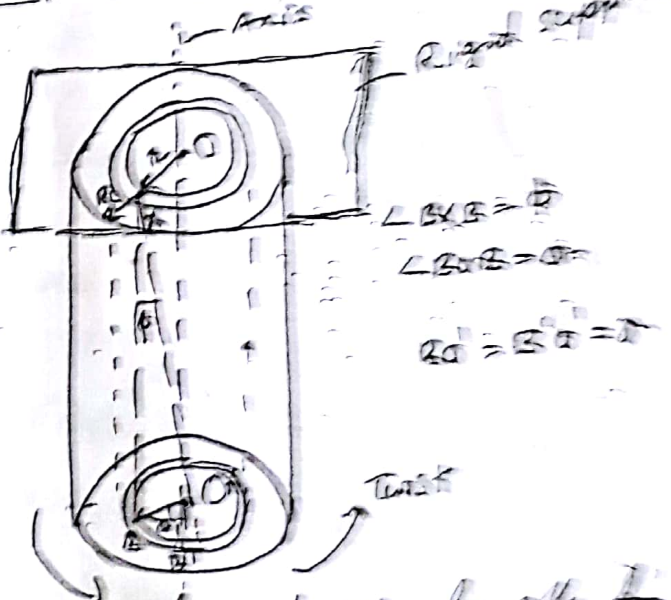
Applications

Beams are used,

1. In the fabrication of trolley ways
2. In the chassis / frame as truck beds
3. In the elevators
4. In the contruction of platforms and bridges
5. As girders in buildings and bridges

Torsion of a Cylinder

Expression for Torsion of a Cylindrical Rod



Consider a long cylindrical rod of length L and radius R, rigidly fixed at its upper end.

Let OO' be its axis

We can imagine the cylindrical rod to be made of thin concentric layers each of thickness 'dr'.

If the rod is twisted at its lower end, then the concentric layers slide one over the other. This movement is zero at the fixed end and gradually increases along the downward direction. Consider one concentric circular layer of radius 'x' and thickness dx.

A point 'A' on the top remains fixed and a point 'B' at its bottom shifts to B'. $\angle BAB' = \phi$ is the angle of shear. Since ϕ is small we have $BB' = l\phi$. Also if $\angle BOB' = \theta$, then arc length $BB' = x\theta$

$$l\phi = x\theta, \text{ or } \phi = \frac{x\theta}{l} \quad \text{—— (1)}$$

Now, the cross sectional area of the layer under consideration is $2\pi x\,dx$. If F is the shearing force, then the shearing stress T is given by,

$$T = \frac{\text{Force}}{\text{Area}} = \frac{F}{2\pi x\,dx} \quad \text{—— (2)}$$

Shearing force $F = T(2\pi x\,dx)$

If ϕ is the shearing angle, then rigidity modulus

$$n = \frac{\text{Shearing stress}}{\text{Shearing strain}} = \frac{T}{\phi}$$

$$T = n\phi = \frac{nx\theta}{l} \quad \text{(from eq 1)}$$

Substituting T in eq (2),

$$F = \frac{n r \theta}{l}(2\pi r\, dr)$$

$$= \frac{2\pi n\theta}{l} r^2 dr$$

Moment of the force about $OO' = \left(\frac{2\pi n\theta}{l} r^2 dr\right) r$

$$= \frac{2\pi n\theta}{l} r^3 dr$$

This is only for one layer.

Twisting couple acting on the entire cylinder

$$= \int_{r=0}^{R} \frac{2\pi n\theta}{l} r^3 dr$$

$$= \frac{2\pi n\theta}{l}\left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{\pi n R^4\theta}{2l}$$

Couple per unit twist 'C' is given by,

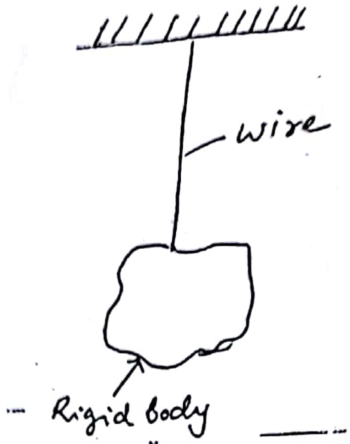
$$C = \frac{\text{total twisting couple}}{\text{angle of twist}}$$

$$= \frac{\pi n R^4\theta/2l}{\theta}$$

$$C = \frac{\pi n R^4}{2l} \quad \text{—— (3)}$$

Torsional pendulum and its period

Consider a straight uniform wire whose one end is fixed on a support and a rigid body is attached to the other end



If the suspended body is rotated slightly and let free, it undergoes regular to and fro turning motion around the wire as its axis. Such type of oscillations are called Torsional oscillations and the set up is called Torsional pendulum.

As per the theory of vibrations the time period of oscillations T for a torsional pendulum is given by,

$$T = 2\pi\sqrt{\frac{I}{C}}$$ where I is the moment of inertia ~~and~~ of the rigid body and C is the couple/unit twist for the wire.

Applications

1. Moment of inertia of irregular bodies can be found.
2. The ~~rigidity~~ rigidity modulus of a material can be found.

FIBERS

Optical fibers are light guides used in optical communications as wave guides. They are able to guide visible and infrared light over long distances.

Construction

Optical fiber

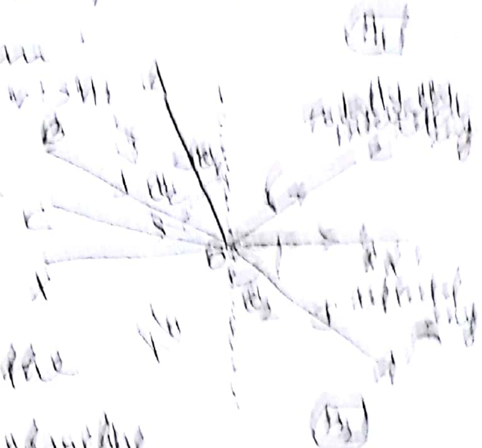
An optical fiber has two parts and it is cylindrical in shape. The inner part is made of glass or plastic and is called core. The outer part surrounding the core is called cladding. The refractive index of cladding is less than that of core but it is made up of the same material. The cladding is enclosed in a polyurethane jacket which safeguards the fiber against chemical reaction with surroundings. Many such fibers are grouped to form a cable.

Total Internal Reflection

The principle based on which the optical fiber works is total internal reflection. Total internal reflection has a special importance. Some light energy is always lost during reflections that occur at the surface of the mirror, or a highly polished mirror surface. But in case of total internal reflection there is absolutely no absorption of light energy at the reflecting surface. The entire incident energy is returned along the reflected light. Hence it is called total internal reflection.

Because of no loss of energy during reflection the optical fibers are able to sustain the light signal transmission over very long distances in spite of infinite number of reflections that occur within the fiber.

Consider a ray AO travelling in a medium of refractive index n_1. Let XX' be the boundary of this medium separating it from another medium of lower refractive index 'n_2'.



Let the incident ray AO make an angle θ_1 with the normal in the medium of refractive index n_1. As this ray is refracted in to the medium of refractive index n_2 it bends away from the normal since $n_1 > n_2$. If θ_2 is the angle made by the refracted ray with the normal, then $\theta_2 > \theta_1$. If θ_1 is increased then for a certain value of $\theta_1 = \theta_c$ called critical angle $\theta_2 = 90°$ i.e. the refracted ray will graze along the boundary of separation along OX' which ...

incident ray is along BO. For any angle of incidence θ_1 which is greater than θ_c, the incident ray like OC always gets reflected back into the medium as per the laws of reflection.

For refraction, the Snell's law is

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

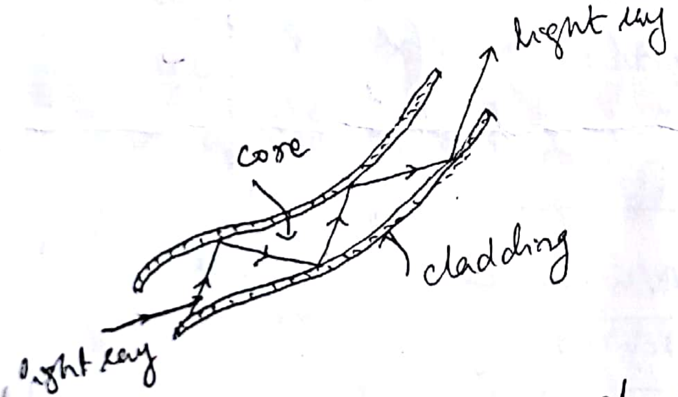
For total internal reflection,

$$\theta_1 = \theta_c, \quad \theta_2 = 90^\circ$$

$$\therefore n_1 \sin\theta_c = n_2 \sin 90 = n_2 \quad (\because \sin 90 = 1)$$

$$\boxed{\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)}$$

Propagation mechanism (or working principle)



Fiber as a wave guide

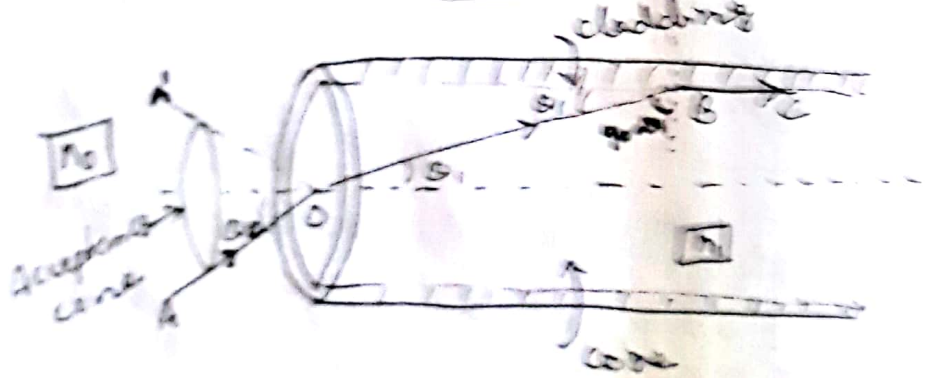
A wave guide is a tubular structure through which energy could be guided in the form of waves. Since light waves are guided through a fiber, it is called light guide. It is also called fiber wave guide. The guiding mechanism can be described as follows.

In any optical fiber, the refractive index of cladding is always lesser than that of for its core. The light signal which enters in to the core can strike the interface of the core and cladding only at large angles of incidence because of the ray geometry shown in fig. The signal undergoes reflection after reflection at the points wherever it is incident on the interface. Since each reflection is a total internal reflection, the signal sustains its strength. Thus the optical fiber functions as a wave guide.

The propagation of light continues as long as the fiber is not bent too sharply. Since at the sharp bends, the light fails to undergo total internal reflection and the signal strength drops drastically. Hence care is taken to avoid very sharp bends in the fiber.

Ray Propagation in the Fiber

* Angle of Acceptance & Numerical Aperture



Let us consider the case of a ray which suffers critical incidence at the core cladding interface.

The ray to begin with travels along AO entering into the core at an angle θ_0 to the fiber axis. Then it is refracted along OB

at ~~my~~ angle θ_1 in the core and further (c)
proceeds to fall at critical angle of incidence
$(90-\theta_1)$ at B on the interface between core
and cladding. Since the angle of incidence
is the ~~cri~~ critical angle, the ray is refracted at 90°
to the normal drawn to the interface ie it
grazes along BC.

Any ray that enters in to the core at an angle of incidence less than θ_0 will have refractive ~~index~~ angle less than θ_1 because of which its angle of incidence $= 90 - \theta_1$ at the interface, will become greater than the critical angle of incidence and hence undergo total internal reflection.

Any ray that enters at an angle greater than θ_0 at O, will incident at the interface at an angle less than the critical angle because of which it gets refracted in to the cladding region and emerges in to the surroundings and thus be lost.

Now if OA is rotated around the fiber axis keeping θ_0 same, then it describes a conical surface. We can therefore say that only those rays which are passing in to the fiber within this cone will only be totally internally reflected. The angle θ_0 is called the wave guide

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acceptance angle and $\sin\theta_0$ is called the numerical aperture (N.A) of the fiber.

Condition for Propagation

Let n_0, n_1 and n_2 be the refractive indices of surrounding medium, core of the fiber and cladding respectively.

Now for refraction at the point of entry of the ray AO in to the core, ~~the~~ applying Snell's law that $n_0 \sin\theta_0 = n_1 \sin\theta_1$ — ①

At the pt. B on the interface.

the angle of incidence $= 90 - \theta_1$

Again according to Snell's law,

$$n_1 \sin(90-\theta_1) = n_2 \sin 90$$

$$n_1 \cos\theta_1 = n_2$$

$$\cos\theta_1 = \frac{n_2}{n_1} \quad \text{— ②}$$

Eqn ① becomes, $\sin\theta_0 = \frac{n_1}{n_0} \sin\theta_1$

$$= \frac{n_1}{n_0}\left(\sqrt{1-\cos^2\theta_1}\right)$$

Substitute $\cos\theta_1$ from eqn ②,

$$\sin\theta_0 = \frac{n_1}{n_0}\sqrt{1-\frac{n_2^2}{n_1^2}} = \frac{\sqrt{n_1^2-n_2^2}}{n_0}$$

If $n_0 = 1$, $\sin\theta_0 = \sqrt{n_1^2-n_2^2}$.

$$N.A = \sqrt{n_1^2-n_2^2}.$$

If θ_1 is the angle of incidence of an incident ray, then the ray will propagate if $\theta_1 < \theta_0$

$$\sin\theta_1 < \sin\theta_0$$

$$\sin\theta_1 < \sqrt{n_1^2-n_2^2}$$

$$\sin\theta_1 < N.A$$

This is the condition for propagation.

Fractional Index Change (Δ)

The fractional index change Δ is the ratio of the ~~di~~ refractive index difference between the core and cladding to the refractive index of core of an optical fiber.

$$\Delta = \frac{(n_1 - n_2)}{n_1} \quad \cdots (1)$$

Relation between N.A and Δ

From eqn (1), $(n_1 - n_2) = n_1 \Delta$

$$NA = \sqrt{n_1^2 - n_2^2}$$

$$= \sqrt{(n_1 + n_2)(n_1 - n_2)}$$

$$= \sqrt{(n_1 + n_2) n_1 \Delta}$$

Since $n_1 \approx n_2$, $(n_1 + n_2) = 2n_1$

$$N.A = \sqrt{2 n_1^2 \Delta}$$

$$N.A = n_1 \sqrt{2\Delta}$$

Mode of propagation

The application of [illegible] shows that, out of the light that enters the core within the wave [illegible] the core within the [illegible] angle, only the light [illegible] certain number of modes [illegible] for propagation in the fiber [illegible] represented as though they [illegible] the figures. Thus a single [illegible] is shown in single mode fiber and so on.

V- number

The number of modes supported for propagation in the fiber is determined by a parameter called V-number. If the surrounding medium is air then the V-number is given by.

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}.$$

d is the core diameter

n_1 is the refractive index of the core

n_2 is the refractive index of cladding

λ is the wavelength of light propagating in the fiber

$$V = \frac{\pi d}{\lambda}(NA)$$

If the fiber is surrounded by a medium of refractive index n_0, then the expression is

$$V = \frac{\pi d}{\lambda}\frac{\sqrt{n_1^2 - n_2^2}}{n_0}$$

For $V >> 1$ the number of modes supported by the fiber is given by, number of modes $\cong \frac{V^2}{2}$.

Types of optical Fibers

The optical fibers are classified under 3 categories

(a) single mode fiber

(b) step index multimode fiber

(c) graded index multimode fiber

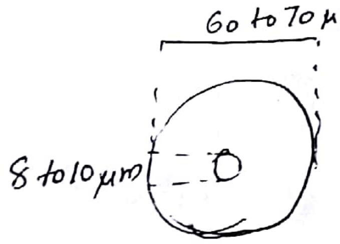
This classification is done depending on the refractive index profile and the no. of modes.

5

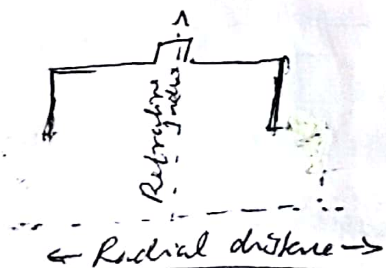
(4) Single mode fiber

A single mode fiber has a core material of uniform refractive index value. Cladding also has a material of uniform index but of lesser value. This results in a sudden increase in the value of refractive index from cladding to core. Thus its refractive index profile takes the shape of a step.

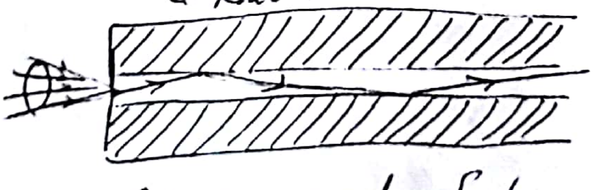
(The curve which represents the variation of refractive index with respect to the radial distance from the axis of the fiber is called refractive index profile.) The diameter of core is about 8 to 10 μm less than that of cladding 60 to 70 μm. Because of its narrow core, it can guide a single mode as shown in figure.



Geometry



Refractive index profile.



Ray propagation.

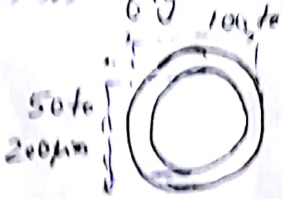
Single mode fiber

Hence it is called single mode fiber.

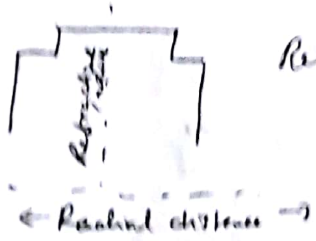
Single mode fibers are most extensively used ones and they constitute 80% of all the fibers that are manufactured in the world today. They need lasers as the source of light. It is less expensive but very difficult to splice them. (Splicing means joining the strands of one optical fiber to the strands of another.) They are used in submarine cable system.

(b) Step Index multimode fiber

The geometry of a step ~~mode~~ index multimode fiber is as shown in fig. Its construction is similar to that of a single mode fiber, but its core has a much larger diameter. Because of this it will be able to support propagation of large number of modes as shown in fig.



Geometry.



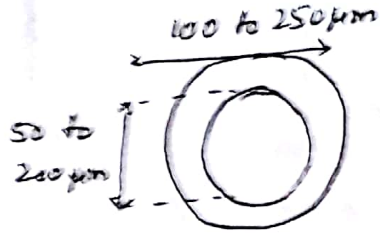
Refractive Index Profile

Ray Propagation

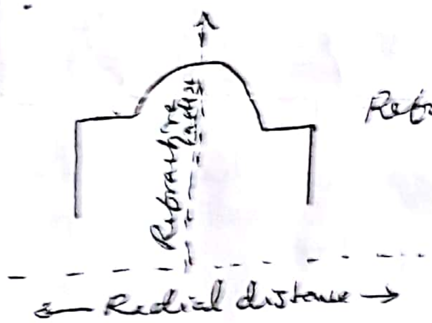
STEP-INDEX MULTIMODE FIBER

Its refractive index profile is also similar to that of a single mode fiber but with a larger plane region for the core. ~~They~~ It needs laser or an LED as source of light. It is the least expensive of all. Its application is in data links which has lower bandwidth requirements.

(c) Graded-Index Multimode Fiber

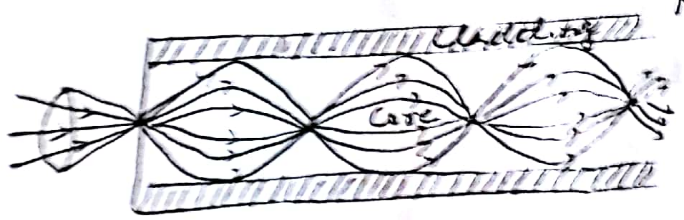


Geometry



Refractive Index Profile

Ray propagation



GRADED - INDEX MULTIMODE FIBER

Graded index multimode fiber is denoted as GRIN. The geometry of the GRIN multimode fiber is same as that of step index multimode fiber. ~~Its core~~ material has a special feature that

~~that~~ ~~Its~~ The refractive index value of core decreases in the radially outward direction from the axis. and becomes equal to that of the cladding at the interface. But the refractive index of the cladding remains uniform. Either a laser or LED can be the source for the GRIN multimode fiber. It is the most expensive of all. Its splicing is difficult. Its application is in the telephone trunk between central offices.

Attenuation

Attenuation is the loss of power suffered by the optical signal as it propagates through the fiber. It is also called the fiber loss.

The processes ~~through~~ which attenuation takesplace are (1) absorption (2) scattering and (3) radiation losses.

Absorption losses

In this case the loss of signal power occurs due to absorption of photons associated with the signal. Photons are ~~absorbed by~~ absorbed by

(1) ~~Absorption by~~ Impurities ~~in the silica glass~~

(2) Intrinsic absorption.

Absorption by Impurities

During signal propagation when photons interact with ~~these~~ impurities, ~~the electrons~~ like iron, chromium etc. the electrons in the impurities absorb the photons and get excited to higher energy level. Later these electrons give up their absorbed energy in the form of heat or light. The re-emission of light energy is of no use since it will usually be in a different wavelength or in different phase with respect to the signal. Hence it is a loss.

The other impurity which would cause of absorption loss is OH (hydroxy) ion, which enters in to the fiber constitution at the time of fiber fabrication

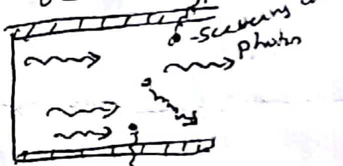
Intrinsic Absorption

The absorption that takes place in the fiber material assuming that there are no impurities in it and that the material is free of all inhomogeneities is called intrinsic absorption

2. Scattering losses

(a) Rayleigh scattering

When the signal travels in the fiber, the photons may be scattered because of sharp changes in refractive index values inside the glass over distances that are small compared to wavelength of light. The sharp variation in refractive index value is because of the structural inhomogeneity. The structural inhomogeneities are set in to the glass during its solidification from molten state. This scattering is same as Rayleigh scattering. Rayleigh scattering occurs when a light wave travels through a medium having scattering objects whose dimensions are smaller than the wavelength. Thus it becomes a loss. The inhomogeneities in glass can not be removed by any processing techniques.



(c) Others.

There are defects in the fiber in the form of trapped gas bubbles, unreacted starting materials, and some crystallized region in the glass. But these are negligible compared to Rayleigh scattering.

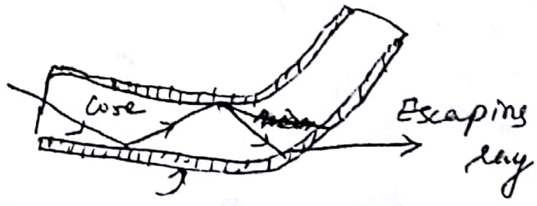
3. Radiation losses.

Radiation losses occur due to

(a) macroscopic and (b) microscopic bends.

Macroscopic bends

They are the bends with radii much larger compared to the fiber diameter. These occurs while wrapping



Radiation leaks.

the fiber on a spool or turning it around a corner the loss will be negligible for small bends but increases until it reaches a critical radius of curvature. Any bend less than this value makes the losses suddenly become extremely large and there is absence of total internal reflection.

Microscopic bends

This type of bends are small scale fluctuations in the linearity of the fiber axis. They occurs due to non-uniformities in the manufacturing of the fiber or by non-uniform lateral pressures created ~~during the~~ during the cabling of the fiber. The microbends cause irregular reflections and some of them leak through the fiber

Attenuation coefficient

When light travels in a material medium there will always be a loss in its intensity with distance travelled. Such a loss of intensity obeys a law called Lambert's law.

As per Lambert's law the rate of decrease of intensity of light with distance travelled in a homogeneous medium is proportional to the initial intensity. ie, if 'P' is the initial intensity and L is the distance propagated in the medium, then

$$-\frac{dP}{dL} \propto P$$ (−ve sign indicates that it is a decrement)

$$-\frac{dP}{dL} = \alpha P \quad \text{—— (1)}$$

where α is a constant called attenuation coefficient. It is also called fiber loss.

From eqn (1), $\frac{dP}{P} = -\alpha\, dL$.

Integrating, $\int \frac{dP}{P} = -\alpha \int dL$ —— (2)

Consider an optical fiber of length L;
if P_{in} is initial intensity and Pout is the intensity of light received as output. Then eqn (2) becomes

$$\int_{P_{in}}^{P_{out}} \frac{dP}{P} = -\alpha \int_0^L dL, \quad [\ln P]_{P_{in}}^{P_{out}} = -\alpha [L]_0^L$$

$$\ln\left(\frac{P_{out}}{P_{in}}\right) = -\alpha L, \quad \alpha = -\frac{1}{L}\ln\left(\frac{P_{out}}{P_{in}}\right) \quad \text{—— (3)}$$

The unit for attenuation is Bel. Variation ~~of~~ in ~~sound~~ intensity is dealt in common logarithm ie, base is taken as ~~log~~10, the equation for 'α' is,

$$\alpha = -\frac{1}{L}\log_{10}\left[\frac{P_{out}}{P_{in}}\right] \text{ Bel/unit length.} \quad \text{—— (4)}$$

In optical fiber technology 'α' should express in terms of decibel/kilometer. (dB/km) and unit Bel. is replaced by 10dB. 1 Bel = 10 decibel. The length is expressed in km. Eqn (4) becomes,

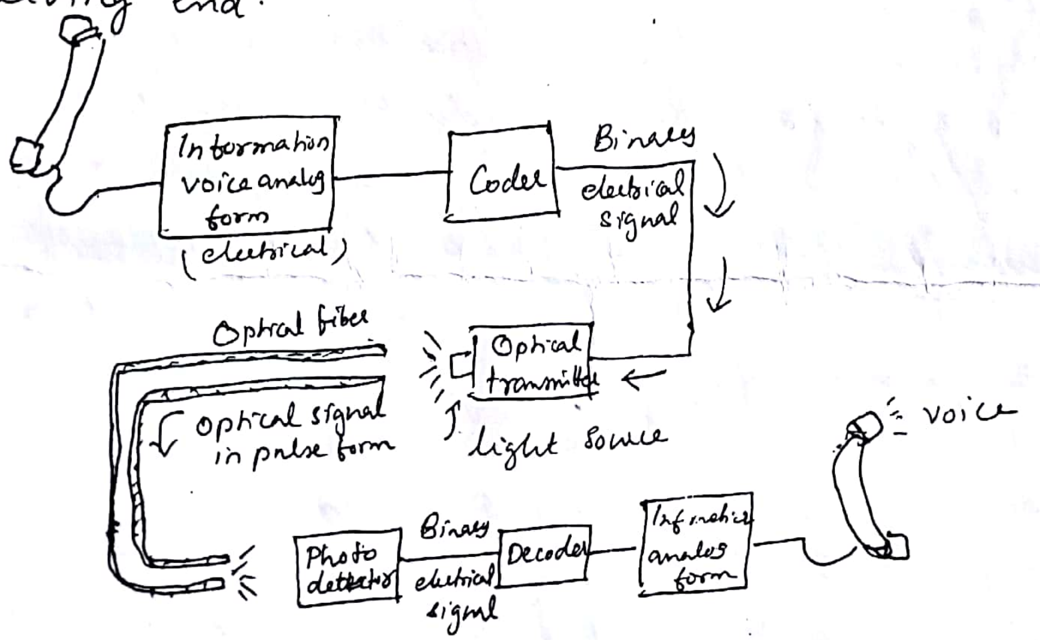
$$\alpha = -\frac{1}{L}\log_{10}\left(\frac{P_{out}}{P_{in}}\right) \times 10\,\text{dB/km}$$

$$\alpha = -\frac{10}{L}\log_{10}\left(\frac{P_{out}}{P_{in}}\right) \text{dB/km.}$$

* Applications of Fiber Optics

Basics of point to point communication system using optical Fibers

Optical Fiber communication is the transmission of information by propagation of optical signal through optical fibers. This involves the conversion of electrical signal to optical signal at the transmitting end, and the conversion of optical signal back to electrical signal at the receiving end.



Typical point to point fiber optic communication system

In a typical point to point communication system, we have analog information such as voice of a telephone user. The voice gives rise to electrical signals in analog form coming out of the transmitter section of the telephone. The analog signal is converted to binary data with the help of an electronic system called codes. The binary data comes out as a streams of electrical pulses from the coder. These electrical pulses are converted to pulses of optical power in a unit called optical transmitter. ~~From~~ These optical pulses are fed in to the fiber.

Out of the incident light which is funnelled in to the ~~fiber~~ core, only certain modes will be sustained for propagation within the fiber by means of total internal reflection. As it propagates, the signal is subjected to both attenuation and delay distortion. (Delay distortion is the reduction in the quality of signal because of spreading of pulses with time). Because of this, the pulses starts overlapping in time.

At this stage a repeater is needed in the ~~transmi~~ transmission path. An optical repeater consists of a receiver and a transmitter. The receiver section converts the optical signal in to corresponding electrical signal. The electrical signal is sent to an optical transmitter section where the electrical signal is again converted back to optical signal, and then fed back into the optical fiber line.

Finally at the receiving end, the optical signal is fed in to a ~~poto~~ photodetector where it is transformed in to pulses of electric current which is then fed in to decoder which converts the binary data stream into analog signal which will be the same information such as voice, which was at the transmitting end.

<u>Other Applications</u> 1. Sensing device, 2. Data link 3. Local area network.

Optical fibers can be used ~~as~~ sensing devices to sense pressure, etc.
Data link is a communication over a distance which is much shorter than that required for a telecommunication trunk.